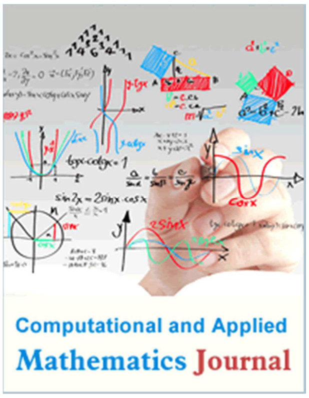
Computational and Applied Mathematics Journal
2015; 1(1): 7-11
Published online January 20, 2015 (http://www.aascit.org/journal/camj)

American Association for Science and Technology



Keywords
Symmetric Bordered Diagonal Matrices,
Inverse Eigenvalue Problems

Received: December 08, 2014
Revised: January 08, 2015
Accepted: January 09, 2015

The inverse eigenvalue problem for some special kind of matrices (I)

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Citation
A. M. Nazari, M. Alibakhshi. The Inverse Eigenvalue Problem for Some Special Kind of Matrices (I). *Computational and Applied Mathematics Journal.* Vol. 1, No. 1, 2015, pp. 7-11.

Abstract
In recent paper [1] (Juang Peng, Xi-Yan Hu. Lei Zang) two inverse eigenvalue problems are solved and in the order article [2] (Hubert Pickmann, Juan Egana, Ricardo L. Soto), a correction, for one of the problems stated in the first article, has been presented as well. In this article, according to the article [2], a solution which is different from the one in the article [1] has been presented for one of the problems which are in the article [1]. The matrix solution in the article [1] and the one which is presented by us, in the main diagonal, are similar, but instead of first column and row, we valued second column and row, furthermore other element of the matrix are considered null.

1. Introduction

In recent paper [1], an inverse eigenvalue problem is solved, a part of which, considering

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)},$$

finds an $n \times n$ matrix B_n, such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of B_j respectively for all $j = 1,2,3,\ldots,n$, where B_j to denote the $j \times j$ leading principal submatrix of B_n, in which B_n is as below:

$$B_n = \begin{pmatrix} a_1 & b_1 & b_2 & b_3 & \ldots & b_{n-1} \\ b_1 & a_2 & 0 & 0 & \ldots & 0 \\ b_2 & 0 & a_3 & 0 & \ldots & 0 \\ b_3 & 0 & 0 & a_4 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ b_{n-1} & 0 & 0 & 0 & \ldots & a_n \end{pmatrix}$$

where a_i are distinct for all $i = 1,2,3,\ldots,n$ and all b_i are positive.

Then consider the following matrix:

$$A_n = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \ldots & 0 \\ b_1 & a_2 & b_2 & b_3 & \ldots & b_{n-1} \\ 0 & b_2 & a_3 & 0 & \ldots & 0 \\ 0 & b_3 & 0 & a_4 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & b_{n-1} & 0 & 0 & \ldots & a_n \end{pmatrix} \quad (1)$$

where a_i are distinct for all $i = 1,2,\ldots,n$ and all b_i are positive. Throughout this paper, we use A_n to denote a special kind of matrices defined as in (1) and A_j to denote the $j \times j$ leading principal submatrix of A_n.

In this paper we, like paper [1], construct a matrix A_n under the following condition:

For $2n-1$ given real numbers $\lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)}$,

we find an an $n \times n$ matrix A_n, such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of A_j respectively for all $j = 1,2,3,\dots,n$.

One of the main problems of the theory of matrices is inverse eigenvalue problem that in recent decades has been of interest to mathematicians, for example inverse eigenvalue problem of nonnegative matrices [4,5,6] or inverse eigenvalue problem of tridiagonal, Jacobi matrices and distance matrices [7,8,9].

In this paper we try to solve a special inverse eigenvalue problem as [1].

2. Properties of the Matrix A_n

Similar paper [1] we assume later on, $b_0 = 1$ and let $\varphi_j(\lambda) = \det(\lambda I_j - A_j)$ and $\varphi_0(\lambda) = 1$.

Lemma 1. For a given matrix A_n, the sequence$\{\varphi_j(\lambda)\}$ satisfies the the following recurrence relation

$$\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - b_{j-1}^{2} \prod_{i=1, i\neq 2}^{j-1} (\lambda - a_i)$$

$$j = 3,4,\dots,n. \tag{2}$$

Proof. By expanding determinant on column or row 2 we can verify the result easily.

Lemma 2. The characteristic polynomial sequence $\{\varphi_j(\lambda)\}$ have some properties of a Sturm sequence, satisfying the following properties:

1) All roots of $\varphi_n(x)$ are real and simple.

2) roots of $\varphi_{j-1}(x)$ and $\varphi_{j+1}(x)$ are distinct and if ξ is a root of $\varphi_j(x)$, then $\varphi_{j+1}(\xi).\ \varphi_{j-1}(\xi)$ <0.

3) $\varphi_0(x)$ has no real root.

Proof. In order to prove this lemma, first we prove that roots are simple, real and satisfy in the following inequality:

$$x_1^{(i)} < x_1^{(i-1)} < \cdots x_{i-1}^{(i-1)} < x_i^{(i)}.$$

By Induction on i, for $i = 1$, we have $\varphi_1(x) = x - a_1$, if $x - a_1 = 0$ then $x = a_1$ and this root is real and simple.

For = 2 ,

$$\begin{aligned}\varphi_2(x) &= (x-a_1)(x-a_2) - b_1^2 \\ &= x^2 - (a_1+a_2)x + a_1a_2 - b_1^2 = 0\end{aligned}$$

Then

$$x_1, x_2 = \frac{(a_1+a_2) \pm \sqrt{(a_1+a_2)^2 - 4a_1a_2 + 4b_1^2}}{2}$$

The roots of $\varphi_2(x)$ are distinct, since if $(a_1+a_2)^2 - 4a_1a_2 + 4b_1^2 = 0$, t hen $(a_1-a_2)^2 = -4b_1^2$ and $a_1 \neq a_2$ which yields contradiction.

The roots of $\varphi_1(x)$ and $\varphi_2(x)$ also are distinct because if a_1 is a root of $\varphi_2(x)$, then we have:

$$\varphi_2(a_1) = (a_1 - a_1)(a_1 - a_2) - b_1^2 \to b_1 = 0$$

which is simple.

Now we show that if ξ_1 and ξ_2 are roots of $\varphi_2(x)$, then $\xi_1 < a_1 < \xi_2$.

To determine the sign of $\varphi_2(x)$ we have the following table:

x	$-\infty$		ξ_1		ξ_2	∞
$\varphi_2(x)$	$+$		0	$-$	0	$+$

Since $\varphi_2(a_1) = -b_1^2 < 0$, then $\xi_1 < a_1 < \xi_2$.

Now assume hypothesis satisfies for $\varphi_j(x)$ where $j \leq i$, as

$$\varphi_{j-1}(x) = x^{j-1} + \cdots$$

Coefficient of the largest power of x is positive then for $x < x_1^{(i-1)}$ the polynomial $\varphi_{i-1}(x)$ near $-\infty$ Equals $(-1)^{i+1}$ or $(-1)^{i-1}$ then table of sign of $\varphi_{i-1}(x)$ is:

x	$-\infty$	$x_1^{(i-1)}$		$\cdots$		$x_{i-1}^{(i-1)}$	∞
$\varphi_{i-1}(x)$	-1^{i+1}	0	-1^{i+2}	$\cdots$	-1^{2i-1}	0	-1^{2i}

By hypothesis of induction, $\varphi_i(x)$ has i distinct roots, which put between roots of $\varphi_{i-1}(x)$, then we have

x	$x_1^{(i-1)}$	$x_1^{(i-1)}$	$\cdots$	$x_k^{(i)}$	$\cdots$	$x_{i-1}^{(i-1)}$	$x_i^{(i)}$
$\varphi_{i-1}(x)$	$(-1)^{i+1}$	0	$\cdots$	$(-1)^{i+k}$	$\cdots$	0	$(-1)^{2i}$

If $x_k^{(i)}$ is a root of $\varphi_i(x)$ then

$$0 = \left(x_k^{(i)} - a_i\right)\varphi_{i-1}\left(x_k^{(i)}\right) - b_{i-1}^2\left(x_k^{(i)} - a_1\right)\cdots\left(x_k^{(i)} - a_{i-1}\right) \tag{3}$$

And

$$\varphi_{i+1}\left(x_k^{(i)}\right) = \left(x_k^{(i)} - a_{i+1}\right)\varphi_i\left(x_k^{(i)}\right) - b_i^2\left(x_k^{(i)} - a_1\right)\cdots\left(x_k^{(i)} - a_i\right) = -b_i^2\left(x_k^{(i)} - a_1\right)\cdots\left(x_k^{(i)} - a_i\right) \tag{4}$$

From (3) and (4) we have

$$\varphi_{i-1}\left(x_k^{(i)}\right) = \frac{b_{i-1}^2\left(x_k^{(i)} - a_1\right)\cdots\left(x_k^{(i)} - a_{i-1}\right)}{\left(x_k^{(i)} - a_i\right)} \times \frac{\left(x_k^{(i)} - a_i\right)b_i^2}{\left(x_k^{(i)} - a_i\right)b_i^2} = \frac{-\varphi_{i+1}\left(x_k^{(i)}\right)b_{i-1}^2}{\left(x_k^{(i)} - a_i\right)^2 b_i^2}$$

consequently $\varphi_{i-1}\left(x_k^{(i)}\right)$ and $\varphi_{i+1}\left(x_k^{(i)}\right)$ have opposite signs.

Now since φ_{i-1} and φ_{i-1}have opposite signs, the table of $\varphi_{i+1}(x)$sign in root of $\varphi_i(x)$ is:

x	$-\infty$	$x_1^{(i)}$	$x_2^{(i)}$	...	$x_i^{(i)}$	∞
$\varphi_{i+1}(x)$	$(-1)^{i+1}$	$(-1)^{i+2}$	$(-1)^{i+3}$	...	$(-1)^{2i+1}$	$(-1)^{2i+2}$

and notice that the sign of $\varphi_{i+1}(x)$ in interval $(-\infty, x_1^{(i)})$ and $(x_i^{(i)}, \infty)$ changes and by Bolzano-Weierstrass theorem, $\varphi_{i+1}(x)$ in both interval has root.

Then we have

$$x_1^{(i+1)} < x_1^{(i)} < \cdots < x_i^{(i)} < x_{i+1}^{(i+1)}.$$

According what we mentioned above all $\varphi_i(x)$ has simple roots and since in i intervals the sign of each φ_i changes and also have i roots, then all roots are real. Since $\varphi_0 = 1$, considering what we said above $\{ \varphi_i \}$ have some properties of a Sturm sequence.

Lemma 3. Assume

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)},$$

be real numbers, where $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of A_j , then we have

$$\lambda_1^{(j)} < a_i < \lambda_j^{(j)}, for\ j = 2, \cdots, n. i = 1,2, \cdots, j$$

Proof. By Induction on, if $j = 2$, since $\varphi_2(\lambda) = (\lambda - a_1)(\lambda - a_2) - b_1^2$, then $\varphi_2(a_1) = \varphi_2(a_2) = -b_1^2 < 0$, with respect to sign of $\varphi_2(\lambda)$ between $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ we have

$$\lambda_1^{(2)} < a_1, a_2 < \lambda_2^{(2)}$$

Now let $\lambda_1^{(j-1)} < a_i < \lambda_{j-1}^{(j-1)}$, for $= 1,2, \cdots, j-1$, we show that $\lambda_1^{(j)} < a_i < \lambda_j^{(j)}, i = 1, \cdots, j$ since for $i = 1, \cdots, j-1$ we have

$$\lambda_1^{(j)} < \lambda_1^{(j-1)} < a_i < \lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}$$

It is enough to prove that $\lambda_1^{(j)} < a_j < \lambda_j^{(j)}$. The sign of $\varphi_j(\lambda)$ in interval $(-\infty, \lambda_1^{(j)})$ is $(-1)^j$and in $(\lambda_1^{(j)}, \lambda_2^{(j)})$ will be $(-1)^{j-1}$ and $\varphi_j(\lambda)$ has negative sign in $(\lambda_{j-1}^{(j)}, \lambda_j^{(j)})$ and positive sign in $\left(\lambda_j^{(j)}, +\infty\right)$.

Applying lemma 1, we have:

$$0 = \varphi_j\left(\lambda_1^{(j)}\right) = \left(\lambda_1^{(j)} - a_j\right)\varphi_{j-1}\left(\lambda_1^{(j)}\right) - b_{j-1}^2\left(\lambda_1^{(j)} - a_1\right)\left(\lambda_1^{(j)} - a_3\right)\cdots\left(\lambda_1^{(j)} - a_{j-1}\right) \quad (5)$$

In (5), the right hand side expression has sign $(-1)^{j-1}$ in which the first expression $\varphi_{j-1}\left(\lambda_1^{(j)}\right)$ has sign $(-1)^{j-1}$, if $(-1)^{j-1}$ is root of $\varphi_j(\lambda)$, this will be necessary for $(\lambda_1^{(j)}, a_j)$ to have negative sign. This means

$$\lambda_1^{(j)} < a_j$$

if we repeat this reason for $\varphi_j\left(\lambda_j^{(j)}\right)$, then we will have $a_j < \lambda_j^{(j)}$, and this proves our claim.

Corollary 1. If $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal zeros of $\varphi_j(\lambda)$ respectively, then

1) for $\mu < \lambda_1^{(j)}$, we have $(-1)^j\varphi_j(\mu) > 0$,

2) for $\mu > \lambda_j^{(j)}$, we have $\varphi_j(\mu) > 0,\ j = 1,2, \cdots, n.$

3. Existence and Uniqueness

Theorem 1. (existence and uniqueness of matrix A_n)

Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for $j = 1,2, \cdots, n$ be real and satisfy in the following relation:

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)}$$

Then there exist the unique matrix A_n in form (1) with $a_i \neq a_j (i, j = 1,2, \ldots, n)$ and $b_i > 0$, where $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are minimal and maximal eigenvalues problem of A_j respectively.

If

$$\lambda_1^{(2)} + \lambda_2^{(2)} \neq 2\lambda_1^{(1)} \quad (6)$$

and

$$\frac{\lambda_{j-1}^{(j-1)} - \lambda_j^{(j)}}{\lambda_{j-1}^{(j-1)} - \lambda_1^{(j)}} < \frac{\varphi_{j-1}(\lambda_1^{(j)}) \prod_{i=1, i\neq 2}^{j-1} (\lambda_j^{(j)} - a_i)}{\varphi_{j-1}(\lambda_j^{(j)}) \prod_{i=1, i\neq 2}^{j-1} (\lambda_1^{(j)} - a_i)} \quad (7)$$

for $j = 3,4, \ldots, n$ or

$$\frac{\lambda_1^{(j-1)} - \lambda_j^{(j)}}{\lambda_1^{(j-1)} - \lambda_1^{(j)}} > \frac{\varphi_{j-1}(\lambda_1^{(j)}) \prod_{i=1, i\neq 2}^{j-1} (\lambda_j^{(j)} - a_i)}{\varphi_{j-1}(\lambda_j^{(j)}) \prod_{i=1, i\neq 2}^{j-1} (\lambda_1^{(j)} - a_i)} \quad (8)$$

for $= 3,4, \ldots, n$, and if we define

$u_{j=}\varphi_{j-1}\left(\lambda_1^{(j)}\right) \prod_{i=1, i\neq 2}^{j-1}\left(\lambda_j^{(j)} - a_i\right)$ and

$v_{j=}\varphi_{j-1}\left(\lambda_j^{(j)}\right) \prod_{i=1, i\neq 2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)$ for $j = 3,4, \ldots, n$, and

$$h_j = \varphi_{j-1}\left(\lambda_1^{(j)}\right) \prod_{i=1, i\neq 2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - \varphi_{j-1}\left(\lambda_j^{(j)}\right) \prod_{i=1, i\neq 2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)$$

Then we can find a_i , b_j by the following relations:

$a_1 = \lambda^{(1)}_1$, $a_2 = \lambda^{(2)}_2 + \lambda^{(2)}_1 - \lambda^{(1)}_1$, $b_1^2 = \left(\lambda^{(2)}_1 - \lambda^{(1)}_1\right)\left(\lambda^{(1)}_1 - \lambda^{(2)}_2\right)$,

$$a_j = \frac{\lambda^{(j)}_1 u_j - \lambda^{(j)}_j v_j}{h_j} \tag{9}$$

$$b_{j-1}^2 = \frac{\lambda^{(j)}_j - \lambda^{(j)}_1)\varphi_{j-1}\left(\lambda^{(j)}_1\right)\varphi_{j-1}\left(\lambda^{(j)}_j\right)}{h_j}, j = 3,4,\dots,n \tag{10}$$

Proof: At first we prove that a_j and b_{j-1} exist for all j, if we denote:
h_j is the denominator a_j of and b_{j-1}^2, and we prove that it is always nonzero. The sign of $\varphi_{j-1}\left(\lambda^{(j)}_1\right)\prod_{i=1,i\neq 2}^{j-1}\left(\lambda^{(j)}_j - a_i\right)$ is $(-1)^{j-1}$ and since $a_i < \lambda^{(j)}_j$ for $= 1,2,\dots,j$, then $\prod_{i=1,i\neq 2}^{j-1}\left(\lambda^{(j)}_j - a_i\right) > 0$ and the sign of $\varphi_{j-1}\left(\lambda^{(j)}_1\right)$ according to which we proved is $(-1)^{j-1}$.
furthermore the sign of - $\varphi_{j-1}\left(\lambda^{(j)}_1\right)\prod_{i=1,i\neq 2}^{j-1}\left(\lambda^{(j)}_j - a_i\right)$ is$(-1)^{j-1}$ and it is nonzero, then denominator of both terms with same sign and nonzero, is nonzero. Then a_j, b_{j-1}^2 exist. Furthermore

$$b_{j-1}^2 = \frac{\left(\lambda^{(j)}_j - \lambda^{(j)}_1\right)\varphi_{j-1}(\lambda^{(j)}_1)\ \varphi_{j-1}(\lambda^{(j)}_j)}{h_j} \tag{11}$$

In numerator 0f (11) sign of $\left(\lambda^{(j)}_j - \lambda^{(j)}_1\right)$ and $\varphi_{j-1}(\lambda^{(j)}_1)$ is positive and $\varphi_{j-1}(\lambda^{(j)}_1)$ has sign $(-1)^{j-1}$, then the sign of numerator is $(-1)^{j-1}$ and the denominator of this rational expression has sign $(-1)^{j-1}$, therefore $b_{j-1}^2$is positive.

Now we prove that a_i which attained are distinct. From $\lambda^{(2)}_1 + \lambda^{(2)}_2 \neq 2\lambda^{(1)}_1$, we have $\lambda^{(2)}_1 + \lambda^{(2)}_2 - \lambda^{(1)}_1 \neq \lambda^{(1)}_1$ consequently $a_2 \neq a_1$.

Let

Now we explain $j = 3$:

The relation (7) includes

$$\frac{\lambda^{(2)}_2 - \lambda^{(3)}_3}{\lambda^{(2)}_2 - \lambda^{(3)}_1} < \frac{(-1)^2 u_3}{(-1)^2 v_3},$$

since v_3 is negative then $(-1)^2\left(\lambda^{(3)}_1 u_3 - \lambda^{(3)}_3 v_3\right) > \lambda^{(2)}_2(-1)^2(u_3 - v_3)$, finally

$$a_3 = \frac{(-1)^2\left(\lambda^{(3)}_1 u_3 - \lambda^{(3)}_3 v_3\right)}{(-1)^2(u_3 - v_3)} > \lambda^{(2)}_2$$

whereas $\lambda^{(2)}_1 < a_1, a_2 < \lambda^{(2)}_2$, then $a_3 \neq a_1 \neq a_2$.
now we assume a_i for $i = 1,2,\dots,j-1$ are distinct, by relation (7) we have

$$\frac{\lambda^{(j-1)}_{j-1} - \lambda^{(j-1)}_j}{\lambda^{(j-1)}_{j-1} - \lambda^{(j-1)}_1} < \frac{(-1)^{j-1}u_j}{(-1)^{j-1}v_j}$$

note that $(-1)^{j-1}v_j$ is negative, then

$$\frac{(-1)^{j-1}(\lambda^{(j)}_1 u_j - \lambda^{(j)}_j v_j)}{(-1)^{j-1}(u_j - v_j)} > \lambda^{(j-1)}_{j-1}$$

this means $a_j > \lambda^{(j-1)}_{j-1}$ and since $\lambda^{(j-1)}_1 < a_i < \lambda^{(j-1)}_{j-1}$ for $i = 1,\dots,j-1$, then we have

$$a_j \neq a_{j-1} \neq \dots \neq a_1.$$

If we use relation (8) we conclude that $a_j < \lambda^{(j-1)}_1$, in which we take distinct a_i for $i = 1,\dots,j$, then the problem has solution and equivalently the following equations:

$$\varphi_j\left(\lambda^{(j)}_1\right) = 0, \varphi_j\left(\lambda^{(j)}_j\right) = 0$$

which has solution distinct a_i for all $i = 1,\dots,n$ and b_{j-1} satisfying b_{j-1} for all $j = 2,\dots,n$. if problem has solution, then

$$\varphi_1(\lambda^{(1)}_1) = (\lambda^{(1)}_1 - a_1) = 0 \Rightarrow a_1 = \lambda^{(1)}_1,$$

$$\varphi_2(\lambda^{(2)}_1) = (\lambda^{(2)}_1 - a_1)(\lambda^{(2)}_1 - a_2) - b_1^2 = 0,$$

$$\varphi_2(\lambda^{(2)}_2) = (\lambda^{(2)}_2 - a_1)(\lambda^{(2)}_2 - a_2) - b_1^2 = 0, \tag{12}$$

then the simplifying we get:

$$a_2 = \frac{(\lambda^{(2)}_2)^2 - {\lambda^{(2)}_1}^2 + \lambda^{(1)}_1(\lambda^{(2)}_1 - \lambda^{(2)}_2)}{\lambda^{(2)}_2 - \lambda^{(2)}_1} = \lambda^{(2)}_2 + \lambda^{(2)}_2 - \lambda^{(1)}_1$$

with substituting a_2 in (12) we have
$b_1^2 = (\lambda^{(2)}_1 - \lambda^{(1)}_1)(\lambda^{(1)}_1 - \lambda^{(2)}_2)$, and since

$$\varphi_j(\lambda^{(j)}_1) = (\lambda^{(j)}_1 - a_j)\varphi_{j-1}(\lambda^{(j)}_1) - b_{j-1}^2\prod_{i=1,i\neq 2}^{j-1}(\lambda^{(j)}_1 - a_i) = 0, \tag{13}$$

and

$$\varphi_j(\lambda^{(j)}_j) = (\lambda^{(j)}_j - a_j)\varphi_{j-1}(\lambda^{(j)}_j) - b_{j-1}^2\prod_{i=1,i\neq 2}^{j-1}(\lambda^{(j)}_j - a_i) = 0, \tag{14}$$

for $3 < j \leq n$ note that with

$$(13) \times \prod_{i=1,i\neq 2}^{j-1}(\lambda^{(j)}_j - a_i) - (14) \times \prod_{i=1,i\neq 2}^{j-1}(\lambda^{(j)}_1 - a_i)$$

we can find (9) and with

$$(13) \times \varphi_{j-1}(\lambda^{(j)}_j) - (14) \times \varphi_{j-1}(\lambda^{(j)}_1)$$

we find (10). Finally uniqueness matrix A_n by (9) and (10) is trivial.

Example 1

For given 7 real numbers

$$\lambda^{(1)}_1 = 6, \lambda^{(2)}_1 = 4, \lambda^{(2)}_2 = 7, \lambda^{(3)}_1 = 3, \lambda^{(3)}_3 = 9, \lambda^{(4)}_1 = 1, \lambda^{(4)}_4 = 14,$$

Finding a 4×4 matrix A_4 that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalue of its $j \times j$ leading principal submatrix. By applying theorem 1, we get the unique solution

$$A_4 = \begin{pmatrix} 6 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 5 & 2.390457218 & 5.108115212 \\ 0 & 2.390457218 & 7.285714286 & 0 \\ 0 & 5.108115212 & 0 & 10.69666390 \end{pmatrix}$$

From the above matrix A_4 we compute the spectrum of A_j , and get

$\lambda(A_1) = 6, \lambda(A_2) = 4{,}7,$

$\lambda(A_3) = 2.99999990{,}6.285714319{,}8.999999981,$

$\lambda(A_4) = 1.00000001{,}6.166248253, 7.816129917{,}14.00000002.$

those obtained data show that Algorithm is correct.

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