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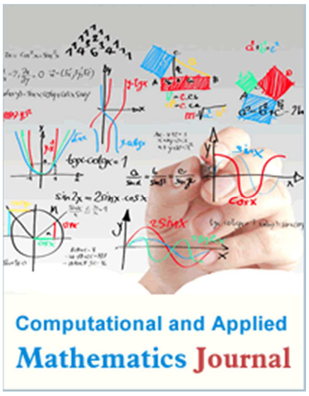

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Saturated Sets in Fuzzy Topological Spaces

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Abstract
The properties of the lattice L, in the L- fuzzy spaces, not only control the properties of the operations on the family of L-fuzzy subsets but also affect some essential properties in L- fuzzy topological spaces. For example, the concepts of limit point and closed subset are important concepts in topology in the theory and in the applications. In ordinary topology every closed subset contains all its limit points. But in the fuzzy case the situation is different: only the saturated fuzzy subset contains all its limit points. It is obtained the formulation of the saturated subsets and it is shown that the family of saturated subsets in fuzzy topological space defines a (saturated) fuzzy topology. In this article, we trace the variation in some properties of some fuzzy topological spaces due to the lattice variation. Most solutions of practical problems in analysis are limit points of some sequences. The results of this article for the saturated set affects the simplicity of finding solutions for many practical problems in fuzzy case.

1. Introduction

The concept of fuzzy subsets has been introduced by Zadeh (1965) [16]. Zadeh defined the fuzzy subset A of X as a function from X into the unit interval of real numbers [0,1]. Goguen (1967) generalized the definition of the fuzzy subset, introducing the L-fuzzy subset of X as a function $A: X \rightarrow L$, where L is a complete and completely distributive complemented lattice [9]. The minimal and maximal elements of L are denoted by 0_L and 1_L respectively (or simply 0 and 1). Since then, the theory of fuzzy mathematics has been developed in many directions and many applications were discovered in a wide variety of fields.

Chang (1968) introduced the concept of fuzzy topology (C- fuzzy topology) [2]. Wong (1973) noted that new concepts of convergence and clustering are needed in order to develop the theory further in this context [15]. Since then, an entensive work on fuzzy topological space has been carried out by many researchers; they have used different lattices for the membership values of elements. In [4], it was given a trail to correct the deviation in the definitions of convergence and clustering in fuzzy topological spaces by studying the $P^*(L)$-fuzzy topological spaces, where $(P^*(L), \cup, \cap, ')$ is a sublattice lattice from the lattice of power set $P(L)$.

In this work, we will demonstrate how the properties of the lattice L not only control the properties of the operations of the family of fuzzy subsets but also control some properties of the fuzzy topological structure on this family. In particular, we will trace the variation in the properties of the limit points and closed sets in fuzzy topological spaces resulting from the lattice variation. It is studied the (saturated) fuzzy subsets which contains all its limit points and obtained its formulation in the fuzzy topological spaces I^X and $P^*(L)^X$.

2. Preliminaries

2.1. In this Paper, We Shall Consider the Following Two Lattices

The first lattice is the closed unit interval $[0,1]$ of real numbers with the usual order and it is denoted by I. The elements of this lattice satisfy the inequality $r \wedge s \neq 0$; for every nonzero numbers r, s. The complementary operation is defined by $' : I \to I;\ r' = 1 - r$, for every $r \in I$. This is an important lattice, which is widely used in the literature.

The operations on the family of fuzzy subsets I^X are defined as follows: Let A, B are I-fuzzy subsets of X, then:

(i) $A \leq B$ if $A(x) \leq B(x)$; for all $x \in X$.

(ii) $(A \wedge B)(x) = A(x) \wedge B(x)$; for all $x \in X$.

(iii) $(A \vee B)(x) = A(x) \vee B(x)$; for all $x \in X$.

(iv) $A^c(x) = (1 - A(x))$; for all $x \in X$.

(v) $\varphi(x) = 0$ and $X(x) = 1$, for all x

The second lattice is denoted by $P^*(I)$ which is a subfamily of the power set $P(I)$ and it is defined by

$$P^*(I) \subset P(I);\ P^*(I) = \{M \subseteq I : 0 \in M\}.$$

The complementary operation $' : P^*(I) \to P^*(I)$ is defined by $M' = (I - M) \cup \{0\}$; for each $M \in P^*(I)$. The algebraic structure $(P^*(L), \cup, \cap, ')$ forms a complete and completely distributive complemented lattice with $\{0\}$ as the smallest element and I as the greatest element. In the lattice $P^*(I)$, $\alpha' \wedge \alpha = 0$; for all $\alpha \in P^*(I)$, where α' is the complement of α. The importance of this lattice is that it generates a family of fuzzy subsets $P^*(I)^X$; the properties of its algebraic operations are very close to the properties of the algebraic operations on the subsets in the ordinary case. This lattice was used in the articles [4].

The operations on the $P^*(I)$-fuzzy subsets of X are defined as follows: Let A, B are $P^*(I)$-fuzzy subsets of X, then:

(a) $A \subseteq B$ if $A(x) \subseteq B(x)$; for all $x \in X$.

(b) $(A \cap B)(x) = A(x) \cap B(x)$; for all $x \in X$.

(c) $(A \cup B)(x) = A(x) \cup B(x)$; for all $x \in X$.

(d) $A^c(x) = (L - A(x)) \cup \{0\}$; for all $x \in X$.

(e) $A^c \cap A = \varphi$, and $A^c \cup A = X$.

Remarks 2.1.1.

(1) It's worth to noting that $P^*(\{0,1\}) = \{\{0\}, \{0,1\}\}$, which is generated by the lattice $L = \{0,1\}$, is isomorphic to the lattice L.

(2) On the first lattice $I = [0,1]$ one can define different complementary operations, as shown in Example (3.1).

(3) The supremum and infimum operations on the family of fuzzy subsets $P^*(L)^X$; for arbitrary lattice L, are coincides with ordinary union and ordinary intersection operations on the sets, consequently, the operations on $P^*(L)^X$ satisfy the same mentioned properties of the operations on $P^*(I)^X$.

(4) The difference operation on the family of $P^*(I)$-fuzzy subsets ($P^*(L)$-fuzzy subsets) is defined as follows: $(A - B)(x) = A(x) - B(x)$; $x \in X$.

2.2. Fuzzy Point

For any lattice L, the fuzzy point is defined as follows:

Definition (2.2.1) [15]. Let L be a lattice and X be a nonempty set. A fuzzy subset H of X is called a fuzzy point, if

$H(x) = 0$; for all $x \neq x_0$ and $H(x_0) = r$; $0_L < r \leq 1_L$ and is denoted by $H = H(x_0, r)$.

It follows that $H(x_0, r) = H(y_0, s)$ iff $x_0 = y_0$ and $r = s$. Therefore, the fuzzy point $H(x_0, r)$ is different from the fuzzy point $H(x_0, s)$ for every $r \neq s$.

2.2.1. Properties of the Family of Fuzzy Points $\mathcal{H}_x = \{H(x, r); r \in L\}$ Relative the Given Lattices

I- For the first lattice $L = [0,1]$ the family $\mathcal{H}_x$ has the following properties:

(i) $H(x, s) \leq H(x, r)$; if $s \leq r$,

(ii) $H(x, s) \wedge H(x, r) = H(x, s \wedge r)$, $H(x, s) \vee H(x, r) = H(x, s \vee r)$,

(iii) If $0 < r < 1$ and r' is the complement of r, then $H(x, r) \vee H(x, r') \neq H(x, 1), H(x, r) \wedge H(x, r') \neq 0$.

Therefore, for every nonzero $r \in [0,1]$, there exists a fuzzy element $H(x_0; s)$ contained in $H(x_0, r)$ by taking $s < r$; for example $s = \frac{r}{2}$. The family $\mathcal{H}_x$ has no disjoint fuzzy points and it is linearly ordered.

II- For the second lattice $L = P^*(I)$ the family $\mathcal{H}_x$ has the following properties:

(i) $H(x, \alpha) \subset H(x, \beta)$; if $\alpha \subset \beta$,

(ii) $H(x, \alpha) \cap H(x, \beta) = H(x, \alpha \cap \beta)$ and $H(x, \alpha) \cup H(x, \beta) = H\ (x, \alpha \cup \beta)$,

(iii) There are disjoint fuzzy points, for example: $H(x, \alpha) \cap H(x, \alpha') = 0_X$; $\alpha \neq \{0\}$,

(iv) This family is not linearly ordered.

The fuzzy point $H(x, \alpha)$ is contained in the fuzzy subset A, if $\alpha \subset A(x)$, and it is not contained in A, if $\alpha \supset A(x)$ and $\alpha \neq A(x)$. But the fuzzy point $H(x, \alpha)$; $\alpha \supset A(x)$ contains the fuzzy point $H(x, A(x))$. In this lattice one can easily show that: $H(x_0, \alpha) \cap H(x_0, \beta) = \varphi$; if $\alpha \cap \beta = \{0\}$; $\alpha, \beta \in P^*(I)$. In particular, there are the following cases:

$$H\left(x_0\ , \left[0, \frac{1}{3}\right]\right) \cap H\left(x_0\ , \{0\} \cup \left(\frac{1}{3}, 1\right]\right) = \varphi; \text{ and}$$

$$H\left(x_0\ , \left[0, \frac{1}{3}\right]\right) \cap H\left(x_0\ , \{0\} \cup \left[\frac{2}{3}, 1\right]\right) = \varphi$$

while

$$H\left(x_0\ , \left[0, \frac{2}{3}\right]\right) \cap H\left(x_0\ , \{0\} \cup \left[\frac{1}{3}, 1\right]\right) \neq \{0\}.$$

The fuzzy points $H(x_0, \alpha)$, $H(x_0, \beta)$ must be regarded as different points for $\alpha \neq \beta$, even if they have the same part x_0.

If the lattice $L = P^*(I)$, the family of fuzzy points $\mathcal{H}_x = \{H(x, \{0, r\}); r \in I\}$ forms a lattice, which is isomorphic to I by the correspondence $r \leftrightarrow H(x, \{0, r\})$.

We say that the fuzzy point $H(x, r)$ is a fuzzy element of the fuzzy subset A if $A(x) = r$.

2.2.2. Limit Point of Fuzzy Subsets

Chang (1968) introduced the definition of topology on fuzzy subsets of the base set X, using the lattice $I = [0,1]$ the unit interval of real numbers [2]. This topology will be called C- fuzzy topology. The C- fuzzy topology on the fuzzy subsets I^X is a collection of fuzzy subsets τ of I^X, which contains X, φ, closed under finite infimum and closed under arbitrary supremum operations. The elements of the topology are called open sets. The definition of the C-fuzzy topology was extended to the family of L- fuzzy subsets L^X [9].

The fuzzy subset is called an ordinary subset if all membership values of all its elements are equal to zero or one. For the fuzzy subsets A, B, it is known that $A \leq B$ if $A(x) \leq B(x)$; for all $x \in X$, and we say that A is contained in B. In the C- fuzzy topology, as in the ordinary case, the closed set is defined as the complement of an open set and the closure of the set is the smallest closed set containing it [1].

The definition of the limit point of a subset in [2] can be formulated into a more suitable form for the fuzzy case as follows:

Definition 2.2.2.1. The fuzzy point $H(x_0, r)$ is called a limit point to the fuzzy subset A in the C- fuzzy topology τ, if for every open fuzzy subset B; for which $H(x_0, r) \wedge B \neq \varphi$, it follows that $A \wedge B \neq \varphi$ and it contains at least one point $H(y, s)$, different from $H(x_0, r)$, in B.

3. Closed Subsets in Fuzzy Topological Spaces

The definition of the closed sets in fuzzy topological space depends on the definition of the complementary operation on the used lattice. Consequently, the closed sets vary as the definition of the complementary operation varies. For example:

Example 3.1. Denote by I_1 the lattice of the closed unit interval of real numbers $I = [0,1]$, with the complementary operation $' : I \to I; r' = 1 - r$. And denote by I_2 the lattice of the closed unit interval of real numbers $I = [0,1]$, with the complementary operation $'' : I \to I$; where $r'' = 1 - 2r$; for all $r \leq (\frac{1}{3})$ and $r'' = \left(\frac{1}{2}\right)(1 - r)$; for all $r \geq (\frac{1}{3})$. Consider the collection of fuzzy subsets,

$$W = \{ \varphi, R, A_r ;\ r \in \{\left(\tfrac{1}{4}\right), \left(\tfrac{1}{3}\right), \left(\tfrac{3}{4}\right)\}\},$$

where R is the set of real numbers and A_r is a fuzzy subset of R defined by the relation $A(x) = r$; for all $x \in R$. It is clear that $\tau_1 = W$ is a C- fuzzy topology on the family of fuzzy subsets I_1^R and $\tau_2 = W$ is a C-fuzzy topology on the family of fuzzy subsets I_2^R. According to the definition of the complementary operations on the lattices I_1 and I_2 the family of closed sets of the topology τ_1 is

$$F_1 = \{ \varphi, R, A_r ; r \in \left\{\left(\tfrac{1}{4}\right), \left(\tfrac{2}{3}\right), \left(\tfrac{3}{4}\right)\right\}\},$$

while the family of closed sets of the topology τ_2 is

$$F_2 = \{ \varphi, R, A_r ; r \in \left\{\left(\tfrac{1}{8}\right), \left(\tfrac{1}{3}\right), \left(\tfrac{1}{2}\right)\right\}\}.$$

Obviously, the closure of the fuzzy subset varies as the complementary operation varies, since the definition of the closure of a set depends on the family of closed subsets. Now, we shall follow up the variation of the relation between the limit points of a fuzzy subset and the closed subsets due to the variation of the used lattice.

Definition 3.1. Let $H(x_0, r)$ be a limit point to the fuzzy subset A in the C- fuzzy topology τ. If for every open fuzzy subset B there exists an element $H(y, s)$ such that $y \neq x_0$, then the limit point $H(x_0, r)$ will be called a limit point of the first type. And if for some x_0, the fuzzy subset $A \wedge B$ does not contain any element $H(y, s)$, for which $y \neq x_0$, then we say that the fuzzy limit point $H(x_0, r)$ is a limit point of the second type.

Example 3. 2. Let $X = \{a, b\}$ and consider the following cases:

(i) If (X, τ_1) is a topological space, where the used lattice is $I = [0,1]$ and $\tau_1 = \{\varphi, X, H\left(a, \frac{1}{3}\right), H\left(a, \frac{3}{4}\right)\}$. In this topology each fuzzy point $H(a, r)$; for every $r > 0$, is a limit point for the fuzzy subset $H(a, \frac{1}{3})$. The limit points in this case are limit points of the second type.

(ii) If (X, τ_2) is a topological space, where the used lattice is $P^*(I)$ and

$$\tau_2 = \{\varphi, X, H\left(a, \left[0, \tfrac{1}{3}\right)\right), H\left(a, \{0\} \cup \left[\tfrac{1}{3}, \tfrac{3}{4}\right)\right), H\left(a, \left[0, \tfrac{3}{4}\right)\right)\},$$

the fuzzy point $H(a, \{0, \frac{2}{3}\})$ is not a limit point for the fuzzy subset $H\left(a, \left[0, \frac{1}{3}\right)\right)$.

Remark 3.1.

(i) If $H(x, r)$ is a fuzzy limit point of the fuzzy subset A in a fuzzy topology τ on I^X, then the fuzzy elements $H(x, s)$; $0 < s \leq 1$ are also limit points for the fuzzy subset A.

(ii) If $H(x, r)$ is a fuzzy limit point of the fuzzy subset A in a fuzzy topology ρ on $P^*(I)^X$, then it does not mean that $H(x, s)$; $r \neq s$ is a limit point for the fuzzy subset A.

The statement "the closed set contains all its limit points", is one of the important properties of the closed sets in the ordinary topology. This statement, in general, is not correct in fuzzy topological spaces.

Example 3.3. In Example (3.2), the closed set $A_{1/8}$, in the topology τ_2, does not contain all its limit points: Let

$\frac{1}{8} < r < \frac{1}{4}$, then A_r is contained in all non empty τ_2 - open sets and $A_r \notin A_{\frac{1}{8}}$. But $A_{\frac{1}{8}}$ intersects with all the non empty τ_2 - open sets in fuzzy points different from A_r, then A_r is a limit point for the fuzzy subset $A_{\frac{1}{8}}$ .This draws our attention to study the properties of the subsets, containing all its limit points, in fuzzy topological spaces.

Definition 3.2. The saturated set in the fuzzy topological space is the set containing all its limit points.

Example 3.4. Using the definition of the saturated set, one can show that the family of saturated sets of the topology τ_1 (or τ_2) on the fuzzy subsets I_1^R (respectively on I_2^R) in Example (3.1) is $\{ \varphi , R \}$ only.

4. Fundamental Theorem of Saturated Sets in Fuzzy Topological Spaces

Consider the following remarks on the closed subsets in the fuzzy topological spaces I^X, $P^*(I)^X$:

(a) Each closed set A in the fuzzy topological space ρ on $P^*(I)^X$ ($P^*(L)^X$) is a saturated set. Since the complement A^c of A is an open set and every fuzzy element $H(x,r) \in A^c$ can`t be a limit point of A due to $A^c \wedge A = 0_L$. Therefore, A contains all its limit points.

(b) Let τ be a fuzzy topological space on I^X . The fuzzy point $H(x,r)$ is a limit point of the closed set A^c, if each open set, intersecting $H(x,r)$, intersects A^c in at least one point different from $H(x,r)$.

Consider the collection of points $X_0 = \{ x_0 ; 0 < A(x_0) < 1\}$:

(i) If the set $X_0 = \varphi$, then the open set A and its complement A^c are ordinary subsets. In this case $A \wedge A^c = 0_X$, and consequently any fuzzy element $H \in A$ can not be a limit point for A^c. Therefore, A^c contains all its limit points and it is a saturated set.

(ii) If $X_0 \neq \varphi$, then each fuzzy point $H = H(x_0, r)$; $x_0 \in X_0$ and $0 < r < 1$ is a limit point of A^c. Since $A(x_0) \wedge r \neq 0$ then, $H(x_0, r) \wedge A \neq 0_X$ and from $A(x_0) \wedge (A(x_0))' \neq 0$ it follows that $A \wedge A^c \neq 0_X$. Therefore, every $B \in \tau$; for which $H(x_0, r) \wedge B \neq 0_X$ satisfies that $B \wedge A^c \neq 0_X$. Consequently, B contains the fuzzy element $H(x_0, s)$, for which $s < \{r, B(x_0), A^c(x_0)\}$. This means that $H(x_0, r)$ (for all $0 < r < 1$) is a limit point for A^c.

(iii) From (ii), it follows that if $X_0 \neq \varphi$ then $H(x_0, 1)$; $x_0 \in X_0$ is a limit point of A^c.

(iv) Consider the collection of points $X_1 = \{ x_0 ; A(x_0) = 1\}$. It is clear that any fuzzy element $H = H(x_0, r)$; $x_0 \in X_1$ and $0 < r \leq 1$ is not contained in A^c. If the point x_0 is not a limit point of A^c, then there exists an open subset B such that $x_0 \in B$ and $B \wedge A^c = 0_X$. Therefore all the elements of the open set B are not limit points of A^c.

From the above discussion in (i), (ii), (iii) and (iv) we get the following theorem on the saturated sets. Therefore, in the fuzzy topologies I^X, $P^*(I)^X$ and $P^*(L)^X$ we have the following result:

Theorem 4.1. (Fundamental theorem on saturated sets).

If τ is a fuzzy topology on the fuzzy subsets I^X, $P^*(I)^X$, then

(i) In any fuzzy topological space on I^X, the family of saturated fuzzy subsets consists of the following $\varphi , X , X - \bigvee_{\substack{B \wedge A^c = \varphi \\ B \in \tau}} (\mathrm{supp}(B))$; $A \in \tau$, which are ordinary subset.

(ii) In any fuzzy topological space on $P^*(I)^X$ ($P^*(L)^X$) , each closed set is a saturated set.

Remarks. In the expression $\bigvee_{\substack{B \wedge A^c = \varphi \\ B \in \tau}} (\mathrm{supp}B)$, the supremum is taken over all open sets $B \in \tau$ for which $B \wedge A^c = \varphi$. Therefore there exists an open fuzzy subset $B(A^c) \in \tau$ such that $B(A^c) = \bigvee_{\substack{B \wedge A^c = \varphi \\ B \in \tau}} B$. Then,

$$\mathrm{supp}B(A^c) = \bigvee_{\substack{B \wedge A^c = \varphi \\ B \in \tau}} (\mathrm{supp}B)$$

Examples of saturated sets.

(1) The family $P^*(I)^X$, where $I = [0,1]$, is a fuzzy topology. Each closed set in this topology is a saturated set. The family of saturated sets, in the fuzzy topological space $P^*(I)^X$, contains ordinary subsets as it contains non ordinary fuzzy subsets.

(2) The family of all fuzzy subsets I^X; $I = [0,1]$ is a fuzzy topology. The family of all saturated sets of this topology consists only of ordinary subsets of X.

The following theorem answered the question: Is there another fuzzy topological spaces other than $P^*(L)^X$ in which each closed subset is a saturated subset?

Theorem 4.2. The fuzzy topological space τ on L^X, in which each closed subset is a saturated subset, satisfies that $A \wedge A^c = \varphi$; for every $A \in \tau$.

Proof. Let (X, τ) be a fuzzy topological space, having the property that each closed subset is a saturated subset. Let $A \in \tau$ be an open subset, then A^c is a saturated subset. Let $H(x_0, r)$ be a fuzzy point of A, then $H(x_0, r)$ is not a limit point of A^c. It follows there exists an open set $(H(x_0, r))$: $O(H(x_0, r)) \wedge A^c = 0_X$. Then, $(\bigvee_{H(x_0,r) \in A} O(H(x_0, r))) \wedge A^c = A \wedge A^c = 0_X$.

5. Saturated Topology

It is interesting to find out the fundamental properties of the family of saturated sets. In the fuzzy topological spaces on I^X. The saturated sets are defined through the open subsets of the form $B(A^c)$; $A \in \tau$. The properties of these subsets are given in the following lemma.

Lemma 5.1. The family of subsets $B(A^c) = \bigvee_{\substack{B \wedge A^c = \varphi \\ B \in \tau}} B$; $A \in \tau$ on the fuzzy topology I^X satisfies the following properties:

(i) $B((\vee_i A_i)^c) = B(\wedge_i A_i^c) = \vee_i B(A_i^c)$, for arbitrary index family i.

(ii) $B((A_1 \wedge A_2)^c) = B(A_1^c \vee A_2^c) = B(A_1^c) \wedge B(A_2^c)$.

Proof. (i) From the definition of $B(A_i^c)$, it follows that $B(A_i^c) \wedge A_i^c = \varphi$; for arbitrary index set i . Then

$B(A_i^c)\wedge(\wedge_j A_j^c) = \varphi$; for all i . Therefore, $B((\wedge_i A_i^c)) \geq \vee_i B(A_i^c)$.

On the other hand, let $H(x_0, r)$ be a fuzzy point. If $H(x_0, r) \notin (\vee_i B(A_i^c))$, then $H(x_0, r)\wedge(\vee_i B(A_i^c)) = \varphi$, therefore $H(x_0, r)\wedge B(A_i^c) = \varphi$; for all i.

This means that in the fuzzy topology τ on the family I^X , we have $x_0 \notin \mathrm{supp}B(A_i^c)$ for all i. It follows that every open fuzzy subset O, for which $x_0 \in \mathrm{supp}O$, satisfies that $O \wedge A_i^c \neq 0_X$; for all i . Since if this relation is valid for some, then we have a contradiction with that $B(A_i^c)$ is a maximal open set satisfies $B(A_i^c)\wedge A_i^c = \varphi$. Therefore, $x_0 \in A_i^c$, for all i and consequently $x_0 \in \wedge_i A_i^c$. But $B(\wedge_i A_i^c)\wedge(\wedge_i A_i^c) = \varphi$, then $x_0 \notin B(\wedge_i A_i^c)$

Therefore, $B(\wedge_i A_i^c) \leq \vee_i B(A_i^c)$. (i) is proved.

(ii) Since $B(A_1^c \vee A_2^c)\wedge(A_1^c \vee A_2^c) = \varphi$, then $B(A_1^c \vee A_2^c)\wedge A_1^c = \varphi$ and $B(A_1^c \vee A_2^c)\wedge A_2^c = \varphi$.

Therefore, $B(A_1^c \vee A_2^c) \leq B(A_1^c)$ and $B(A_1^c \vee A_2^c) \leq B(A_2^c)$. It follows that $B(A_1^c \vee A_2^c) \leq B(A_1^c)\wedge B(A_2^c)$.

Now, since $(B(A_1^c) \wedge B(A_2^c))\wedge(A_1^c \vee A_2^c) = \varphi$, then $B(A_1^c)\wedge B(A_2^c)) \leq B(A_1^c \vee A_2^c)$.

Therefore, $B(A_1^c \vee A_2^c) = B(A_1^c) \wedge B(A_2^c)$. (ii) is proved.

Notation. In the following, we shall using the notation:

$$F(A^c) = X - \vee_{\substack{B\wedge A^c=\varphi \\ B\in\tau}} (\mathrm{supp}B) = X - \mathrm{supp}(B(A^c)); \; A \in \tau,$$

where $B(A^c) = \vee_{\substack{B\wedge A^c=\varphi \\ B\in\tau}} B$.

Since all the closed subsets in any fuzzy topology τ on $P^*(I)^X$ are saturated subsets, then the induced topology by the saturated subsets is the same topology τ. The following theorem gives the important properties of the saturated subsets in the topology defined on I^X.

Theorem 5.1. The family of saturated subsets of the fuzzy topology on I^X satisfies the following properties: If A_i, for every index i are open fuzzy subset, then

(i) $F(A_1^c \vee A_2^c) = F(A_1^c)\vee F(A_2^c)$.

(ii) $F(\wedge_i A_i^c) = \wedge_i F(A_i^c)$, for arbitrary index set $\{i\}$.

Proof.

For the defined fuzzy topology I^X, using the above lemma, we get

$$F(\wedge_i A_i^c) = X - \mathrm{supp}\, B(\wedge_i A_i^c) = X - \mathrm{supp}\vee_i B(A_i^c)$$

$$= X - \vee_i \mathrm{supp}B(A_i^c) = \wedge_i[X - (\mathrm{supp}\, B(A_i^c))] = \wedge_i F(A_i^c)$$

$$F(A_1^c)\vee F(A_2^c) = [X - \mathrm{supp}B(A_1^c)]\vee[X - \mathrm{supp}B(A_2^c)]$$

$$= X - [\mathrm{supp}B(A_1^c)\wedge \mathrm{supp}B(A_2^c)] = X - [\mathrm{supp}(B(A_1^c)\wedge B(A_2^c))]$$

$$= X - \mathrm{supp}(B(A_1^c \vee A_2^c) = F(A_1^c \vee A_2^c).$$

Since the fuzzy subsets in fuzzy topology on I^X satisfies that $\mathrm{supp}\, A \wedge \mathrm{supp}B = \mathrm{supp}(A \vee B)$ due to $r \wedge s \neq 0$; for all nonzero $r, s \in L$. Thus Theorem (5.1) is proved.

The family of saturated subsets generates topology of the complements of the saturated subsets, which is called associated topology to the enveloped subsets.

Theorem 5.2. To each fuzzy topology on I^X there exists an associated topology of the saturated subsets, which are the complements of the saturated subsets.

6. Conclusion

The associated topology to the family of saturated subsets in any topological spaces on $P^*(I)^X$ is identical with the given topology. Moreover, the associated topology to the saturated subsets, in any fuzzy topological spaces on I^X, is a topology whose elements are the complements of the saturated subsets. The associated topology, in this case, is different from the given topology and consists from ordinary subsets only.

Although the article does not contain any applications, but the obtained results improve the methods of obtaining the solutions in fuzzy case. Most solutions of problems in analysis are obtained as a limit point of some sequences. For this we will look for solutions to these problems within the appropriate saturated sets. The choice of the saturated sets affects the simplicity of finding solutions in many practical problems.

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