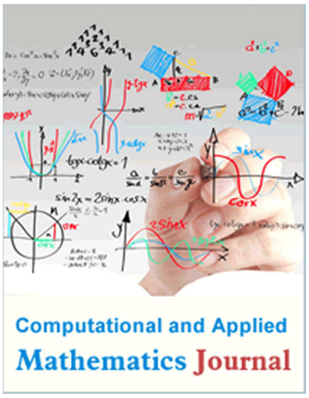
Computational and Applied Mathematics Journal
2015; 1(4): 147-150
Published online May 20, 2015 (http://www.aascit.org/journal/camj)

American Association for Science and Technology



Keywords
Heat Equation,
Boundary Controllability,
Double Fourier Series

Received: March 27, 2015
Revised: April 9, 2015
Accepted: April 10, 2015

A Numerical Method for Controllability of the Heat Equation

Hoda Saffaran[1], Mojtaba Baymani[2]

[1]Department of Mathematics, Islamic Azad University, Quchan Branch, Quchaan, Iran
[2]Department of Mathematics, Quchan University of advanced technology, Quchan, Iran

Email address

h.safaran@yahoo.com (H. Saffaran), m_baymani@qiet.ac.ir (M. Baymani)

Citation

Hoda Saffaran, Mojtaba Baymani. A Numerical Method for Controllability of the Heat Equation. *Computational and Applied Mathematics Journal.* Vol. 1, No. 4, 2015, pp.147-150.

Abstract

In this paper a numerical method based on the double Fourier series is developed for obtaining the solution to boundary controllability of the 1D heat equation. The Fourier series of the solution is written subject to the boundary, initial and final conditions satisfied exactly. Then the Fourier series coefficients are obtained by solving an optimization problem. The details of the method are discussed and the capabilities of the method are illustrated by solving heat problem with different boundary conditions.

1. Introduction

The controllability problems in one-dimensional are known to be solvable since the seventies: we mention to the earlier contributions [1, 2] for some proofs based on spectral arguments. The numerical approximation schemes of boundary control for the heat equation are an important problem. Glowinski et al. [3, 4] devoted to approximate controllability using duality. This is due to the intrinsic ill-posedness of the problem we have just pointed out. For the null boundary case in one dimensional space, we mention the motion planning method introduced in [5] allowing a semi-explicit expression of controlled solutions in term of Gevrey series. This approach has been adapted and numerically developed recently in [6] to obtain inner controls.

In practice, the null control problem is then reduced to the minimization of a dual conjugate function with respect to the final condition of the adjoint state [7, 8, 10].

The theory of the Fourier series involves expansions of arbitrary functions in certain types of trigonometric series. It proves that any periodic function in an interval of time could be represented by the sum of a fundamental and a series of higher orders of harmonic components at frequencies which are integral multiples of the fundamental component. The series establishes a relationship between the function in time and frequency domains. Today, the theory has become the famous 'Fourier series' and it is one of the most important tools for engineers and scientists in many applications.

Theorem: (Convergence of Fourier series)

Let $f(x)$ and $f'(x)$ be piecewise continuous functions on $[-l, l]$ and $f(x)$ be periodic with period $2l$ then f has a Fourier Series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n cos \frac{n\pi x}{l} + b_n sin \frac{n\pi x}{l} \right) = S(x)$$

where

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x)\ cos \frac{n\pi x}{l} dx \ and \ \ b_n = \frac{1}{l} \int_{-l}^{l} f(x)\, sin \frac{n\pi x}{l} dx$$

the Fourier Series converges to f (x) at all points at which f is continuous and to the average value of the left and right limits ($\frac{f(x_-)+f(x_+)}{2}$) at a point of discontinuity of the function f.

Proof: the proof is in [9].

The Fourier series representations extend in a natural way to functions $f(x,t)$ of two real variables x and t over the intervals $-l \le x \le l$ and $-T \le t \le T$, provided f can be represented as a Fourier series in x when t is held constant, and as a Fourier series in t when x is held constant.

The general double Fourier series representation of $f(x,t)$ over the interval $-l \le x \le l$ and $-T \le t \le T$ is given by

$$f(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(a_{ij}cos\frac{i\pi x}{l}cos\frac{j\pi t}{T} + b_{ij}cos\frac{i\pi x}{l}sin\frac{j\pi t}{T})$$

$$+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(c_{ij}sin\frac{i\pi x}{l}cos\frac{j\pi t}{T} + d_{ij}sin\frac{i\pi x}{l}sin\frac{j\pi t}{T})$$

A function $f(x,t)$ that is specified on the interval $0 \le x \le l$ and $0 \le t \le T$ can be represented in terms of a Fourier series on the interval. These series are obtained by first extending the definition of the function to the interval $-l \le x \le l$ and $-T \le t \le T$ in a suitable manner, and then restricting the Fourier series representation of the extended function to the original interval.

2. Description of Method

We are concerned in this work the controllability of the following 1D heat equation:

$$y_t - a(x)y_{xx} = h(x,t) \tag{1}$$

$$y(x,0) = f_0(x), \quad x \in (0,1)$$

$$y(0,t) = g_0(t), \quad t \in (0,T)$$

$$y(x,T) = f_T(x), \quad x \in (0,1)$$

$$y(1,t) = u(t), \quad t \in (0,T)$$

where $y = y(x,t)$ is the state, $f_0, f_T, a \in L^2(0,1), g_0 \in L^2(0,T)$ and $h(x,t)$ are known functions and $u = u(t)$ is the control function which acts on the extreme $x = 1$. We aim at changing the dynamics of the system by acting on the boundary of the domain (0, 1).

To obtain an approximate solution of problem (1), we define

$$z(x,t) = \frac{y(x,t)-A(x,t)}{B(x,t)} \tag{2}$$

where the functions $A(x,t)$ and $B(x,t)$ are given.

Now, we can write the Fourier series of $z(x,t)$ as follow:

$$Z^{(M,N)}(x,t) = \sum_{i=0}^{M}\sum_{j=0}^{N}(a_{ij}cos\frac{i\pi x}{l}cos\frac{j\pi t}{T} + b_{ij}cos\frac{i\pi x}{l}sin\frac{j\pi t}{T}) +$$

$$\sum_{i=0}^{M}\sum_{j=0}^{N}(c_{ij}sin\frac{i\pi x}{l}cos\frac{j\pi t}{T} + d_{ij}sin\frac{i\pi x}{l}sin\frac{j\pi t}{T}) \tag{3}$$

when the numbers M and N are large enough, the Fourier series representation $z^{(M,N)}(x,t)$ is a suitable approximate of $z(x,t)$ and so we have the approximation of $y(x,t)$ in the following form:

$$y^{(M,N)}(x,t) = A(x,t) + B(x,t)\{(\sum_{i=0}^{M}\sum_{j=0}^{N}(a_{ij}cos\frac{i\pi x}{l}cos\frac{j\pi t}{T} + b_{ij}cos\frac{i\pi x}{l}sin\frac{j\pi t}{T}) + \sum_{i=0}^{M}\sum_{j=0}^{N}(c_{ij}sin\frac{i\pi x}{l}cos\frac{j\pi t}{T} + d_{ij}sin\frac{i\pi x}{l}sin\frac{j\pi t}{T})\} \tag{4}$$

Here, the functions $A(x,t)$ and $B(x,t)$ are defined as:

$$A(x,t) = \frac{(T-t)}{T}\left(f_0(x) - f_0(0)\right) + \frac{t}{T}\left(f_T(x) - f_T(0)\right) + g_0(t)$$

$$B(x,t) = xt(T-t).$$

With this choice, the function $y^{(M,N)}(x,t)$ can satisfy the boundary, initial and final conditions. We substitute $y^{(M,N)}(x,t)$ in the first equation of (1) and the Fourier Series coefficients in such a way to bring that the following objective function to be minimized

$$Min_W \sum_{(x_i,t_i)\epsilon D}\{\left(y_i^{(M,N)}(x_i,t_i) - a(x_i)y_{xx}^{(M,N)}(x_i,t_i) - h(x_i,t_i)\right)^2\}, \tag{5}$$

where $D = [0,1]\times[0,T]$ and $W = \{a_{ij}, b_{ij}, c_{ij}, d_{ij}: i = 0, 1, \dots M, j = 0,1,\dots,N\}$.

As the choice (4) the minimization problem (5) is quadratic. When the problem was solved, the Fourier coefficients and thus the approximate solution of problem (1) is obtained.

3. Numerical Results

We now offer some numerical experiment and survey the behavior of the computed control with respect to data, M and N.

Example1.3: As in [8], consider the problem (1) with the following data:

$$f_0(x) = sin(\pi x), h(x,t) = 0, g_0(t) = f_T(x) = 0,$$

We assume that the diffusion a is constant and equal to $a(x) = a_0 = \frac{1}{4}$, $x \in (0,1)$ and take a controllability time equal $T = \frac{1}{2}$. We introduce the notation

$$\Omega=(0,1)\times(0,T), \quad \Gamma=\{1\}\times(0,T)$$

The problem (1) subject to the above data with the various M,N is solved. It is interesting to note that our method was convergence to a steady case for M=N=5 (by 25 independent functions), since the all coefficients of the solution for M and N greater than 5 are equal zero.

Table 1 gives various norms of the solution $y^{(M,N)}(x,t)$ with respect to M, N and h, and clearly suggests the convergence of the approximation.

Table 1. *Numerical results with respect to M=N=5.*

$\Delta x = \Delta t$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{10}$	$\frac{1}{12}$
$\Vert y(x,t)\Vert_{L^2(\Omega)}$	0.2865	0.2977	0.2944	0.2896
$\Vert y(x,t)\Vert_{L^2(\Gamma)}$	0.4555	0.4709	0.4799	0.4633

Figure 2 depicts the corresponding error functions for $M = N = 5,\ h = \frac{1}{6}$ and $\frac{1}{8}$. Also, Figure 3 depicts the corresponding error functions for $M = N = 5, h = \frac{1}{10}$ and $\frac{1}{12}$.

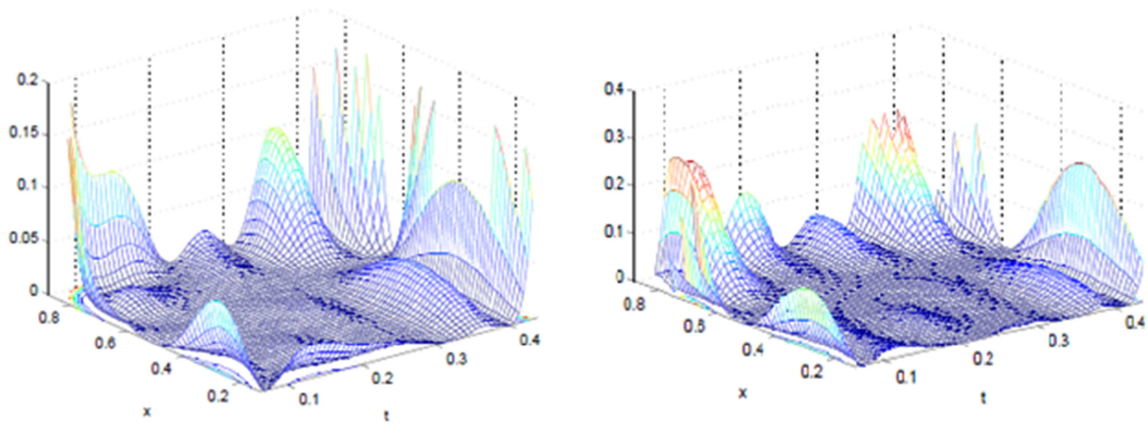


Figure 1. *The error function for M=N=5 and respect to $\Delta x = \Delta t = \frac{1}{16}$ (left), and $\Delta x = \Delta t = \frac{1}{8}$ (right).*

Figure 2 depicts the corresponding solution $y(x,t)$ and the control function u(t). It is also interesting to note that the control obtained is quite regular near $t = T$.

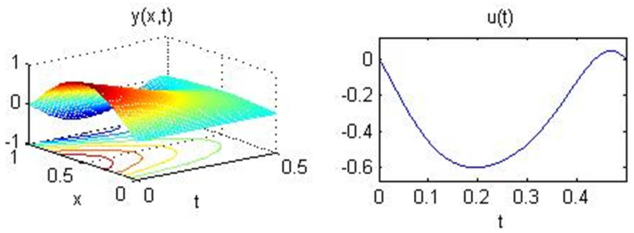


Figure 2. *The solution $y(x,t)$ (left) and the control function u(t) (right).*

Example 2.3: consider the problem (1) with the following data:

$$f_0(x) = sin(\pi x)\,, h(x,t) = 0, g_0(t) = 0,$$

$$f_T(x) = 0.1\, sin(\pi x).$$

We assume that the diffusion a is constant and equal to $a(x) = a_0 = \frac{1}{4} x \in (0,1)$, and take a controllability time equal $T = \frac{1}{2}$.

The problem (1) subject to the above data with the various M, N is solved. It is interesting to note that our method was convergence to a steady case for M=N=5 (use 25 independent functions), since the all coefficients of the solution for M and N greater than 5 are equal zero.

Table 2 gives various norms of the solution $y^{(M,N)}(x,t)$ with respect to M, N and h, and clearly suggests the convergence of the approximation.

Table 2. *Numerical results with respect to M=N=5.*

$\Delta x = \Delta t$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{10}$	$\frac{1}{12}$
$\Vert y(x,t)\Vert_{L^2(\Omega)}$	0.2880	0.2861	0.2834	0.2806
$\Vert y(x,t)\Vert_{L^2(\Gamma)}$	0.2529	0.3014	0.3115	0.3018

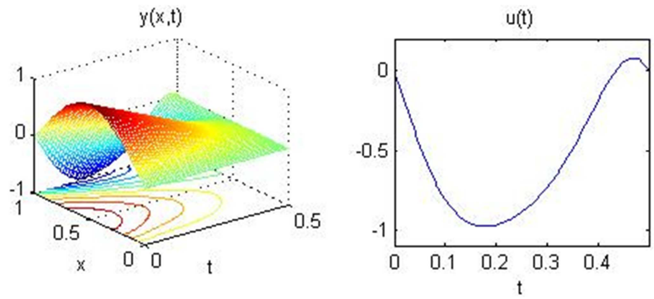


Figure 3. *The solution $y(x,t)$ (left) and the control function u(t) (right).*

4. Conclusion

In this paper a new method based on Fourier series has been applied to find approximate solutions for boundary controllability of heat equations. The solutions via this method are differentiable, closed analytic form easily used in any subsequent calculation. The method here allows us to obtain the solution of control problem starting from randomly sampled data sets and refined it without wasting memory space and therefore reducing the complexity of the problem. If we compare the numerical results, we see that our method has been convergence.

References

[1] H. O. Fattorini and D. L. Russel, Exact controllability theorems for linear parabolic equation in one space dimension, Arch. Rational Mech. 43 (1971) 272-292.

[2] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions*, SIAM Review 20(1978) 639-739.

[3] C. Carthel, R. Glowinski and J. L. Lions, On exact and approximate Boundary Controllability for the heat equation: A numerical approach, J. Optimization, Theory and Applications 82(3), (1994) 429-484.

[4] R. Glowinski, J. L. Lions and J. He, Exact and approximate controllability for distributed parameter systems: A numerical approach Encyclopedia of Mathematics and its Applications, 117. Cambridge University Press, Cambridge, 2008.

[5] B. Laroche, Ph. Martin and P. Rouchon, Motion planing for the heat equation, Int. Journal of Robust and Nonlinear Control 10(2000) 629-643.

[6] A. Munch and E. Zuazua, Numerical approximation of null controls for the heat equation: ill-posedness and remedies, Inverse Problems 26 (2010) no. 8 085018, 39pp.

[7] S. Ervedoza and J. Valein, On the observability of abstract time-discrete linear parabolic equations, Revista Matemática Complutense 23(1) (2010) 163-190.

[8] P. Pedregal, A variational perspective on controllability, Inverse Problems 26 (2010) no. 1, 015004, 17pp.

[9] E. Kreyszig, Advanced engineering mathematics, John Wiley & Sons, Inc. 9th edition (2006).

[10] G. Alessandrini and L. Escauriaza(2008), Null-controllability of one-dimensional parabolic equations, ESAIM Control Optim. Calc. 14(2)(2008), no. 2: 284-293.