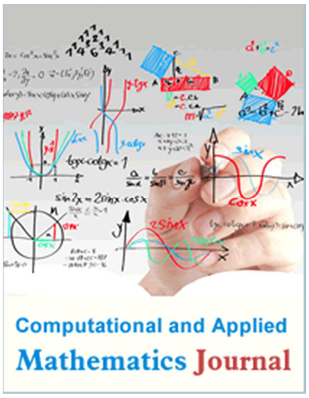
Computational and Applied Mathematics Journal
2015; 1(2): 44-49
Published online April 10, 2015 (http://www.aascit.org/journal/camj)

AASCIT American Association for Science and Technology



Keywords
Black-Scholes Partial Differential Equation, European Call Option, Modified Mellin Transform

Received: March 20, 2015
Revised: March 28, 2015
Accepted: March 29, 2015

On a New Technique for the Solution of the Black-Scholes Partial Differential Equation for European Call Option

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Citation
Sunday Emmanuel Fadugba, Chuma Raphael Nwozo. On a New Technique for the Solution of the Black-Scholes Partial Differential Equation for European Call Option. *Computational and Applied Mathematics Journal.* Vol. 1, No. 2, 2015, pp. 44-49.

Abstract
This paper presents a new technique for the solution of the Black-Scholes partial differential equation for European call option using a method based on the modified Mellin transform. We also used the modified Mellin transform method to determine the price of European call option. The modified Mellin transform method is mutually consistent and agrees with the values of Black-Scholes model as shown in Table 1.

1. Introduction

In finance, an option is a contract which gives the buyer (the owner) the right, but not the obligation, to buy or sell an underlyingasset or instrument at a specified strike price on or before a specified date. The seller has the corresponding obligation to fulfill the transaction – that is to sell or buy – if the buyer (owner) "exercises" the option. The buyer pays a premium to the seller for this right. An option that conveys to the owner the right to buy something at a specific price is referred to as a call; an option that conveys the right of the owner to sell something at a specific price is referred to as a put.

American options allow option holders to exercise the option at any time prior to and including its maturity date, thus increasing the value of the option to the holder relative to European options, which can only be exercised at maturity. The majority of exchange-traded options are American.

European options are options that can only be exercised at the end of its life, at its maturity. They tend to sometimes trade at a discount to its comparable American option. This is because American options allow investors more opportunities to exercise the contract.

Until 1973, the valuation of an option was done by little more than guesswork. Black-Scholes [2] and Merton [5] derived a second order partial differential equation for the value of an option on stocks which then transformed option valuation into a science. Nowadays, the Black-Scholes equation is widely used in the field of financial mathematics. Despite the success of the Black-Scholes model on hedging and pricing contingent claims, Merton [5] noted early that options quoted on the markets differ systematically from their predicted values, which led up to questioning the distributional assumptions based on geometric wiener process.

The Mellin transforms in the theory of option pricing was introduced by Panini and Srivastav [7]. They derived the expression for the free boundary and price of an American perpetual put as the limit of finite lived options. Panini and Srivastav [8] considered the pricing of perpetual options using Mellin transforms. Nwozo and Fadugba [6] considered the Mellin transform method for the valuation of some vanilla power options with non-dividend yield. They extended the Mellin transform method to derive the price of European and American power put options with non-dividend yield. They also derived the fundamental valuation formula known as the Black-Scholes model using the convolution property of the Mellin transform method. Samuelson [9] derived a closed-form expression for the free boundary and price of a perpetual American put option using Mellin transform techniques.

Vasilieva [10] introduced a new method of pricing Multi-options using Mellin transforms and integral equations. Jódar et al [4] considered a new direct method for solving the Black-Scholes equation using the Mellin transforms. F. AlAzemi et al [1] obtained an analytical solution of the Black-Scholes equation for the European and the American put options. In this paper we shall derive a new technique which does not require variables transformation for the solution of homogeneous Black-Scholes partial differential equation for a European call option using the modified Mellin transform proposed by Frontczak and Schöbel [3].

The structure of the paper is organized as described in the following. In the next section we give an overview of the most fundamental ideas and mathematical tools needed for the Mellin transforms. We also present the most relevant properties of the Mellin transforms. Section 3 presents a new technique for the solution of the Black-Scholes partial differential equation for European call option. Section 4 presents numerical experiment. Section 5 concludes the paper.

2. Mellin Transform and Its Fundamental Properties in the Theory of Option Valuation

Let $f(y)$ be a function defined on the positive real axis $t\in(0,\infty)$. The Mellin transformation denoted by M is the operation mapping the function f into the function F defined on the complex plane by the relation

$$M(f(y),v)\equiv F(v)=\int_0^\infty f(y)y^{v-1}dy \tag{1}$$

The function $F(v)$ is called the Mellin transform of f. In general, the integral does exist only for complex values of $v=a+jb$ such that $a\in(a_1,a_2)$, where a_1 and a_2 depend on the function $f(y)$ to transform. This introduces what is called the strip of definition of the Mellin transform that will be denoted by $V(a_1,a_2)$. In some cases, this strip may extend to half-plane $(a_1=-\infty)$ or $(a_2=\infty)$ or to the complex v-plane $(a_1=-\infty)$ and $(a_2=\infty)$.

Conversely, the inversion formula of (1) is defined as

$$M^{-1}(F(v))\equiv f((y),v)=\frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty}F(v)y^{-v}dv \tag{2}$$

where the integration is along a vertical line through $\mathrm{Re}(v)=a$.

Some of the basic fundamental properties of the Mellin transforms are detailed below.

If $f(x)$ is defined on the positive real axis $t\in(0,\infty)$, then the following properties hold.

(a) Shifting Property

$$M\left(y^a f(y);v\right)=\int_0^\infty y^a f(y)y^{1-v}dy = F(v+a) \tag{3}$$

(b) Scaling Property

$$M\left(f(ay;v)\right)=\int_0^\infty f(ay)y^{v-1}dy=a^{-v}F(v) \tag{4}$$

(c) The Mellin Transform of Derivatives

$$M\left(\frac{d^k}{dy^k}f(y);v\right)=(-1)^k(v-k)_kF(v-k),\,(v-k)\in V_f,\,k\in Z^+ \tag{5}$$

where the symbol $(v-k)_k$ is defined for k integer by;

$$(v-k)_k=(v-k)(v-k+1)\ldots(v-1)$$

Equations (3) and (6) can be used in various ways to find the effect of linear combination of differential operator such that $v^k\left(\frac{d}{dv}\right)^m$, k, m integers. The most remarkable results are

$$\begin{cases} M\left(\left(v\frac{d}{dy}\right)^k f(y);v\right)=(-1)^k v^k F(v)\\ M\left(\frac{d^k}{dv^k}v^k f(y);v\right)=(-1)^k(v-k)_k F(v)\\ M\left(v^k\frac{d^k}{dv^k}f(y);v\right)=(-1)^k(v)_k F(v)\end{cases} \tag{6}$$

where $v\in V_f$, k a positive integer and

$v_k = v(v+1)...(v+k-1)$.

3. A New Technique for the Solution of the Black-Scholes Partial Differential Equation for a European Call Option

Let us consider the homogeneous Black-Scholes partial differential equation for a European call $E_c(S_t,t)$ option with the initial and boundary conditions given by

$$\left.\begin{array}{l}\dfrac{\partial E_c(S_t,t)}{\partial t}+\dfrac{\sigma^2 S_t^2}{2}\dfrac{\partial^2 E_c(S_t,t)}{\partial S_t^2}+rS_t\dfrac{\partial E_c(S_t,t)}{\partial S_t}-rE_c(S_t,t)=0,\ S_t\in(0,\infty), t\in[0,T]\\ E_c(S_t,T)=\left(S_t-K\right)^+\\ \lim\limits_{S_t\to 0}E_c(S_t,t)=0\\ \lim\limits_{S_t\to\infty}E_c(S_t,t)=\infty\end{array}\right\}\tag{7}$$

where σ is the volatility, r is a risk-free interest rate, K is called the strike price and T is the maturity date . It is a known fact that the partial differential equation in (7) has a closed form solution obtained after several change of variables and solving certain related diffusion equations. This procedure is not applicable in the vector framework where $E_c(S_t,t)$ is a vector and σ , r are matrices.

The modified Mellin transform for the price of European call option is defined as

$$M(E_c(S_t,t),-v)=\hat{E}_c(v,t)=\int_0^\infty E_c(S_t,t)S_t^{-1-v}dS_t \tag{8}$$

and the inversion formula for the modified Mellin transform is given by

$$M^{-1}(\hat{E}_c(v,t))=E_c((S_t,t),-v)=\frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty}\hat{E}_c(v,t)S_t^v dv \tag{9}$$

To utilize the modified Mellin transform and the conditions that guarantee its existence, we assume that $E_c(S_t,t)$ is bounded of polynomial degree when $S_t\to 0$ and $S_t\to\infty$ i.e.

$$E_c(S_t,t)=\begin{cases}O(S_t^{a_1}),\ S_t\to 0\\ O(S_t^{a_2}),\ S_t\to\infty\end{cases}\tag{10}$$

for any $u\in C$ on $-a_1<R(u)<-a_2$ where $(-a_1,-a_2)$ is called fundamental strip.

Taking the modified Mellin transform of the Black-Scholes partial differential equation for a European call option in (7), we have that

$$M\left(\frac{\partial E_c(S_t,t)}{\partial t}+\frac{\sigma^2 S_t^2}{2}\frac{\partial^2 E_c(S_t,t)}{\partial S_t^2}+rS_t\frac{\partial E_c(S_t,t)}{\partial S_t}\right)=M\left(rE_c(S_t,t)\right)\tag{11}$$

Using the properties of the Mellin transforms, we have

$$\left.\begin{array}{l}M\left(\dfrac{\partial E_c(S_t,t)}{\partial t}\right)=\dfrac{d}{dt}\hat{E}_c(v,t)\\ M\left(\dfrac{\sigma^2 S_t^2}{2}\dfrac{\partial^2 E_c(S_t,t)}{\partial S_t^2}\right)=\dfrac{\sigma^2}{2}(v^2-v)\hat{E}_c(v,t)\\ M\left(rS_t\dfrac{\partial E_c(S_t,t)}{\partial S_t}\right)=rv\hat{E}_c(v,t)\\ M\left(rE_c(S_t,t)\right)=r\hat{E}_c(v,t)\end{array}\right\}\tag{12}$$

Substituting (12) into (11) and simplifying further yields

$$\frac{d\hat{E}_c(v,t)}{dt}=-\left(\frac{v^2\sigma^2}{2}-\left(\frac{\sigma^2}{2}-r\right)v-r\right)\hat{E}_c(v,t),\quad t\in[0,T]\tag{13}$$

Integrating (13) yields

$$\hat{E}_c(v,t)=A(v)\exp\left(-\left(\frac{v^2\sigma^2}{2}-\left(\frac{\sigma^2}{2}-r\right)v-r\right)\right)t\tag{14}$$

Setting $\varphi(v)=\left(\frac{v^2\sigma^2}{2}-\left(\frac{\sigma^2}{2}-r\right)v-r\right)$, then (14) becomes

$$\hat{E}_c(v,t)=A(v)\exp\left(-\varphi(v)\right)t\tag{15}$$

where $A(v)$ is a constant of integration to be determined and it is defined as

$$A(v)=\psi(v,t)\exp(\varphi(v)T)\tag{16}$$

$\psi(v,t)$ can be obtained by taking the modified Mellin transform of the initial condition of the form

$$E_c(S_t,T)=\theta(S_t)=(S-K)^+ \tag{17}$$

then we have

$$\psi(v,t)=M(\theta(S_t),-v)=\int_0^\infty (S_t-K)^+ S_t^{-v-1}dS_t=\frac{K^{1-v}}{v(v-1)} \tag{18}$$

Using equations (15), (16) and (18), we have that

$$\hat{E}_c(v,t)=\frac{K^{1-v}}{v(v-1)}\exp(\varphi(v)(T-t)) \tag{19}$$

The modified Mellin inversion of (19) is obtained as

$$M^{-1}(\hat{E}_c(v,t))=E_c(S_t,t)=\frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty}\frac{K^{1-v}}{v(v-1)}\exp(\varphi(v)(T-t))S_t^v dv \tag{20}$$

Next we want to prove that the expression (20) is a solution of the Black-Scholes partial differential equation for a European call option given by (7). Let us assume that

$$v=m+jn \quad \Rightarrow dv=jdn \tag{21}$$

Substituting (21) into (20) yields

$$E_c(S_t,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}S_t^{(m+jn)}\exp(\varphi(m+jn)(T-t))dn \tag{22}$$

Substituting $v=m+jn$ into $\varphi(v)=\left(\frac{v^2\sigma^2}{2}-\left(\frac{\sigma^2}{2}-r\right)v-r\right)$, then

$$\left.\begin{aligned}\varphi(m+jn)&=\left(\frac{(m+jn)^2\sigma^2}{2}-\left(\frac{\sigma^2}{2}-r\right)(m+jn)-r\right)\\&=\left(\frac{\sigma^2m^2}{2}-\frac{\sigma^2n^2}{2}-\frac{\sigma^2m}{2}+rm-r\right)+j\left(mn\sigma^2-\frac{\sigma^2n}{2}+rn\right)\end{aligned}\right\} \tag{23}$$

Since $E_c(S_t,t)$ is Mellin transformable and continuous, setting $t=T$ then (22) becomes

$$E_c(S_t,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}S_t^{(m+jn)}dn \tag{24}$$

Equation (22) is well defined and satisfies (24).
Using the definition of the Mellin transform, then

$$\left|\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}\right|\le M(m)=\int|f(s)|s^{m-1}ds \quad \forall n\in R \tag{25}$$

and for $t\in[0,T)$ we have that

$$\left\{\begin{aligned}&\int_{-\infty}^{\infty}\left|\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}\right|\left|S_t^{(m+jn)}\right|\left|\exp(\varphi(m+jn)(T-t))\right|dn\\&\le M(m)S_T^m\exp\left(\left(\frac{\sigma^2m^2}{2}-\frac{\sigma^2n^2}{2}-\frac{\sigma^2m}{2}+rm-r\right)(T-t)\right)\int_{-\infty}^{\infty}\exp\left(\left(\frac{-\sigma^2m^2}{2}\right)(T-t)\right)dm\end{aligned}\right. \tag{26}$$

Using the differentiation theorem of parameter integrals and the fact that

$$\int_{-\infty}^{\infty}n^j\exp\left(-\frac{\sigma^2}{2}\right)(T-t)dn<\infty,\quad j=0,1,2,...,t\in[0,T) \tag{27}$$

Then it follows that upon differentiation of (22), we have that

$$\left.\begin{aligned}\frac{\partial E_c(S_t,t)}{\partial t} &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}\varphi(m+jn)S_t^{(m+jn)}\exp(\varphi(m+jn)(T-t))dn \\ \frac{\partial E_c(S_t,t)}{\partial S_t} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}(m+jn)S_t^{(m+jn-1)}\exp(\varphi(m+jn)(T-t))dn\end{aligned}\right\} \tag{28}$$

$$\frac{\partial^2 E_c(S_t,t)}{\partial S_t^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}(m+jn)(m+jn-1)S_t^{(m+jn-2)}\exp(\varphi(m+jn)(T-t))dn \tag{29}$$

Substituting (22), (28) and (29) into the Black-Scholes partial differential equation for a European call option given by (7), we have

$$\left.\begin{aligned}&\frac{\partial E_c(S_t,t)}{\partial t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 E_c(S_t,t)}{\partial S_t^2} + rS_t\frac{\partial E_c(S_t,t)}{\partial S_t} - rE_c(S_t,t) \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}\varphi(m+jn)S_t^{(m+jn)}\exp(\varphi(m+jn)(T-t))dn \\ &+\frac{\sigma^2 S_t^2}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}(m+jn)(m+jn-1)S_t^{(m+jn-2)}\exp(\varphi(m+jn)(T-t))dn + \\ &rS_t\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}(m+jn)S_t^{(m+jn-1)}\exp(\varphi(m+jn)(T-t))dn \\ &r\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}S_t^{(m+jn)}\exp(\varphi(m+jn)(T-t))dn \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-\varphi(m+jn)+\frac{\sigma^2 S_t^2}{2}(m+jn)(m+jn-1)S_t^{(n-2)} + rS_t(m+jn)S_t^{(-1)} - r\right) \\ &\quad\times\left(\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}S_t^{(m+jn)}\exp(\varphi(m+jn)(T-t))\right)dn \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-\varphi(m+jn)+\left(\left(\frac{\sigma^2m^2}{2}-\frac{\sigma^2n^2}{2}-\frac{\sigma^2m}{2}+rm-r\right)+j\left(mn\sigma^2-\frac{\sigma^2n}{2}+rn\right)\right)\right) \\ &\quad\times\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}S_t^{(m+jn)}\exp(\varphi(m+jn)(T-t))dn \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-\varphi(m+jn)+\varphi(m+jn)\right)\frac{K^{1-m-jn}}{(m+jn)(m+jn-1)}S_t^{(m+jn)}\exp(\varphi(m+jn)(T-t))dn \\ &= 0\end{aligned}\right\} \tag{30}$$

Hence $E_c(S_t,t)$ defined by (20) is a solution of (9) and the following result is established.

Theorem: Let the price of a European call option denoted by $E_c(S_t,t)$ be Mellin transformable and continuous, then

$$E_c(S_t,t) = \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty}\frac{K^{1-v}}{v(v-1)}\exp(\varphi(v)(T-t))S_t^v dv$$

for $S_t, t \in [0,T)$ is a solution of the homogeneous Black-Scholes partial differential equation for the European call option of the form

$$\left.\begin{aligned}&\frac{\partial E_c(S_t,t)}{\partial t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 E_c(S_t,t)}{\partial S_t^2} + rS_t\frac{\partial E_c(S_t,t)}{\partial S_t} - rE_c(S_t,t) = 0,\ S_t\in(0,\infty), t\in[0,T] \\ &E_c(S_t,T) = \left(S_t-K\right)^+ \\ &\lim_{S_t\to 0}E_c(S_t,t) = 0 \\ &\lim_{S_t\to\infty}E_c(S_t,t) = \infty\end{aligned}\right\}$$

4. Numerical Experiment

We consider the valuation of European call option using the modified Mellin transform in the context of Black-Scholes model with the following parameters:

$$S_t = 100, 110, 120, K = 100, T = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0,$$

$$r = 0.05, \sigma = 0.2$$

The results generated are shown in the Table below.

Table. *The Comparative Results Analysis of the Black-Scholes Model and the Modified Mellin Transform Method*

T	S	Black-Scholes Model	Modified Mellin Transform Method
0.5	100	6.8887	6.8887
0.5	110	14.0754	14.0754
0.5	120	22.5928	22.5928
1.0	100	10.4506	10.4506
1.0	110	17.6630	17.6630
1.0	120	26.1690	26.1690
1.5	100	13.4429	13.4429
1.5	110	20.7744	20.7744
1.5	120	29.1703	29.1703
2.0	100	16.1268	16.1268
2.0	110	23.5901	23.5901
2.0	120	31.9648	31.9648
2.5	100	18.6033	18.6033
2.5	110	26.1956	26.1956
2.5	120	34.5878	34.5878
3.0	100	20.9244	20.9244
3.0	110	28.6389	28.6389
3.0	120	37.0671	37.0671

5. Conclusion

In this paper we developed a new technique for the solution of the homogeneous Black-Scholes partial differential equation for a European call option using the modified Mellin transform method. The approach used in this paper does not require variables transformation. The above Table demonstrates that the modified Mellin transform method for the valuation of European call option which pays no dividend yield performs very well and agrees with the values of the Black-Scholes model. Also we can see from the Table above that the higher the time to maturity, the higher the values of the transform method.

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