Problems on Foundations of General Relativity

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Abstract

It was generally believed that, in general relativity, the fundamental laws of nature should be invariant or covariant under a general coordinate transformation. In general relativity, the equivalence principle tells us the existence of a local inertial coordinate system and the fundamental laws in the local inertial coordinate system which are the same as those in inertial reference system. Then, after a general

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coordinate transformation, the fundamental laws of nature in arbitrary coordinate system or in arbitrary curved space-time can be obtained. However, through a simple example, we find that, under a general coordinate transformation, basic physical equations in general relativity do not transform covariantly, especially they do not preserve their forms under the transformation from a local inertial coordinate system to a curved space-time. The origination of the violation of the general covariance is then studied, and a general theory on general coordinate transformations is developed. Because of the the existence of the non-homogeneous term, the fundamental laws of nature in arbitrary curved space-time can not be expressed by space-time metric, physical observable and their derivatives. In other words, basic physical equations obtained from the equivalence principle and the principle of general covariance are different from those in general relativity. Both the equivalence principle and the principle of general covariance can not be treated as foundations of general relativity. So, what are the foundations of General Relativity? Such kind of essential problems on General Relativity can be avoided in the physical picture of gravity. Quantum gauge theory of gravity, which is founded in the physics picture of gravity, does not have such kind of fundamental problems.

1 Introduction

In classical theory of gravity, gravity is treated as a kind of physical interactions in flat space-time and obeys inverse square law[1]. In general theory of relativity, gravity is treated as space-time geometry[2, 3]. In order to consider quantum effects of gravitational interactions, various kinds of quantum gravity are studies. In some quantum theories of gravity, gravity is treated as a kind of interactions in space-time. Therefore, there are two different ways to treat gravity, or there are two essentially different points of view on the nature of gravity: one considers gravity as a kind of fundamental physical interactions, and another considers gravity as space-time geometry. For the sake of convenience, we call the former the physics picture of gravity and the latter the geometry picture of gravity.

For some fundamental problems, two pictures of gravity are not consistent. For example, in geometry picture of gravity, there are no physical gravitational interactions in space-time and all gravity effects are only effects of curved space-time, while in physics picture of gravity, gravity is treated as a kind of fundamental physical interactions in space-time and space-time is always kept flat. What is the nature of gravity? In other words, gravity is a kind of fundamental interactions or space-time geometry? Is our space-time essentially flat or curved?

A fundamental theory to describe various kinds of fundamental interactions in nature is gauge theory[4]. Electromagnetic interactions, weak interactions and strong interactions can all be described by gauge theory. Various kinds of unified theories of fundamental interactions are also based on gauge theory, such as unified electroweak theory[5, 6, 7, 8] is a $SU(2)_L \times U(1)_Y$ gauge theory. Now it is generally believed that four kinds of fundamental interactions in nature are all gauge interactions and they can be described by gauge field theory. From theoretical point of view, the principle of local gauge invariance plays a fundamental role in particle's interaction theory.

Gauge theory is so successful in describing fundamental interactions of nature and in unifying different kinds of fundamental interactions, it is natural for physicist to study gravity and to unify gravity with other kinds of fundamental interactions by gauge field theory. In 1918, H. Weyl first use gauge theory to study graivty[9]. Later, gauge gravity is studied extensively[10, 11, 12, 13, 14, 15, 16, 17]. In the traditional gauge treatment of gravity, Lorentz group is localized, and the gravitational fields are not represented by gauge potentials.

Recently, a new quantum gauge theory of gravity, which is perturbatively renormalizable in 4-dimentional space-time, is proposed by N. Wu[18, 19, 20, 21]. In quantum gauge theory of gravity, gravity is treated as a kind of fundamental interactions, so it is formulated in physics picture of gravity. Quantum gauge theory of gravity is based on gauge principle, which is the common nature of all kinds of fundamental interactions in nature. One of the most important advantage of quantum gauge theory of gravity is that four different kinds of fundamental interactions in nature can be unified in a simple and beautiful way[22, 23, 24]. Quantum gauge theory of gravity can be used to explain both quantum phenomena and classical phenomena of gravitational interactions.

However, basic dynamics of gravitational interactions given by different theories of gravity is different. There are two ways to test different kinds of gravity theories. One is to test by experiments. Another way is to check self-consistency of the theory itself. In this paper, the second way is used to test different kinds of gravity theories.

Another fundamental important problem on gravity theory is that what is the symmetry of gravitational interactions. In general relativity, the symmetry of gravitational interactions is general covariance under a general coordinate transformation. In most traditional quantum gravity, the symmetry of gravitational interactions is local Lorentz symmetry. In quantum gauge theory of gravity, the fundamental symmetry is gravitational gauge symmetry. So, different theories have different symmetries. However, the objective gravitational interactions in nature must have definite symmetry. So, what is the symmetry of gravity? The theory with incorrect symmetry can not be an acceptable fundamental theory of gravity. To determine the true symmetry of gravity will help us to determine which is the fundamental theory of gravity.

In this paper, our main spirit is to study the symmetry of gravity. Using a simple example, it can be proved that almost all basic physical equations in general relativity do not transform covariantly under a general coordinate transformation. In this simple model, the transformation rule of basic physical equations can be calculated explicitly. It is found that the transformation rules given by direct calculation are different from those in general relativity. So, what is the origin of such kind of violation? This question is answered by a general theory, which is developed to study the origin of such kind of violation of general covariance under a general coordinate transformation. The violation of general covariance means that both the principle of general covariance and the equivalence principle, which are believed to be *a priori* foundations of general relativity, are no longer the foundations of general relativity. Combining discussions in this paper and results from gauge principle[18, 19, 20, 21], it will be argued that the correct symmetry for gravitational interactions is gravitational gauge symmetry.

2 Foundations of General Relativity

Before we study problems on foundations of general relativity, let's remember what are the foundations of general relativity. It is well known that the foundations of general relativity are two principles: the equivalence principle and the principle of general covariance[2, 3, 25]. Based on these two principles, basic physical equations in arbitrary gravitational field can be obtained.

Equivalence principle states that at every space-time point in an arbitrary gravitational field it is possible to choose a local inertial coordinate system such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation. There are two key points in the physics content of the equivalence principle: the existence of a local inertial coordinate system and the form of the laws of nature in the local inertial coordinate system. According to the equivalence principle, there must exist a local inertial coordinate system at every space-time point in an arbitrary curved space-time, and the forms of the laws of nature in the local inertial coordinate system are the same as those in inertial reference system. It is known that the laws of nature in a inertial reference system are given by special relativity. According to special relativity, basic physical equations in a inertial reference system are

$$\partial_\mu J^\mu = 0, \tag{2.1}$$

$$\frac{\mathrm{d}^2 \xi^\mu}{\mathrm{d}\tau^2} = 0, \tag{2.2}$$

$$\frac{\mathrm{d}A^\mu}{\mathrm{d}\tau} = 0, \tag{2.3}$$

$$\partial_\nu T^{\mu\nu} = 0, \tag{2.4}$$

where J^μ is a conserved current, ξ^μ is the space-time coordinate of a mass point, A^μ is an arbitrary vector, $T^{\mu\nu}$ is energy-momentum tensor and τ is the proper time. Eq.(2.1) is the continuity equation, eq.(2.2) is the equation of motion for a free mass point, eq.(2.3) is the parallel transport equation and eq.(2.4) is the energy-momentum conservation equation. Equivalence principle tells us that these equations hold in a local inertial coordinate system.

But the equivalence principle does not directly tell us what are the forms of the laws of nature in an arbitrary curved space-time. This task is accomplished by the principle of general covariance. The principle of general covariance states that a physical equation holds in a general gravitational field, if two conditions are met: (1) The equation holds in the absence of gravitation; that is, it agrees with the laws of special relativity when the metric tensor $g_{\alpha\beta}$ equals the Minkowski tensor $\eta_{\alpha\beta}$ and when the affine connections $\Gamma^\alpha_{\beta\gamma}$ vanishes. (2) The equation is generally covariant; that is, it preserves its form under a general coordinate transformation. According to the principle of general covariance, the basic physical equations in an arbitrary curved space-time are

$$\nabla_\alpha J^\alpha = 0, \tag{2.5}$$

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = 0, \tag{2.6}$$

$$\frac{\mathrm{D}A^\alpha}{\mathrm{D}\tau} = 0, \tag{2.7}$$

$$\nabla_\beta T^{\alpha\beta} = 0, \tag{2.8}$$

where ∇_α is a covariant derivative, $\frac{\mathrm{D}}{\mathrm{D}\tau}$ is a covariant derivative along the curve $x^\mu(\tau)$ which is defined by

$$\frac{\mathrm{D}A^\alpha}{\mathrm{D}\tau} = \frac{\mathrm{d}A^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}A^\beta. \tag{2.9}$$

It is generally believed that eqs.(2.5) – (2.8) transform covariantly under a general coordinate transformation, so they preserve their forms under a general coordinate transformation; that is, these equations hold in an arbitrary curved space-time.

The key point in the principle of general covariance is that all basic physical equations should be covariant under a general coordinate transformation. It is generally believed that all physical equations, including Einstein field equation, are covariant under a general coordinate transformation, so they have the same form in all possible curved space-times and all possible coordinate systems. If a physical equation does not transform covariantly, it will have different forms in different coordinate systems. In this case, all basic physical equations are space-time dependent, and we can not simple write out an equation which will hold in an arbitrary coordinate system.

Though it was generally believed that Einstein field equation

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -8\pi G_N T_{\alpha\beta} \tag{2.10}$$

and eqs.(2.5) – (2.6) are covariant under a general coordinate transformation, through a simple example, we will show that they are not really covariant under a general coordinate transformation, which will cause serious problems to the foundations of general relativity.

3 A Simple Example

For the sake of convenience, coordinates of flat space-time are denoted by $\xi^\mu = (t, r, \theta, \varphi)$ and coordinates of curved space-time are denoted by $x^\alpha = (t', r', \theta', \varphi')$. Greek indices μ, ν, κ, λ, and so on generally run over the four space-time inertial coordinate labels 0, 1, 2, 3 or t, r, θ, φ; Greek indices α, β, γ, δ and so on generally run over the four coordinate labels in a general coordinate system. The metric $\eta_{\mu\nu}$ of flat space-time has diagonal elements -1, +1, r^2, $r^2\sin^2\theta$. All calculations in this chapter are performed in spherical coordinate system.

Let's discuss a simple coordinates transformation from flat space-time ξ^μ to curved space-time x^α:

$$\mathrm{d}\xi^\mu \to \mathrm{d}x^\alpha = \frac{\partial x^\alpha}{\partial \xi^\mu}\mathrm{d}\xi^\mu, \tag{3.1}$$

$$\frac{\partial}{\partial \xi^\mu} \to \frac{\partial}{\partial x^\alpha} = \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial}{\partial \xi^\mu}, \tag{3.2}$$

where transformation matrices $\frac{\partial x^\alpha}{\partial \xi^\mu}$ and $\frac{\partial \xi^\mu}{\partial x^\alpha}$ are defined by

$$\frac{\partial \xi^\mu}{\partial x^\alpha} = \begin{pmatrix} \sqrt{1-u^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-u^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.3}$$

$$\frac{\partial x^\alpha}{\partial \xi^\mu} = \begin{pmatrix} \frac{1}{\sqrt{1-u^2}} & 0 & 0 & 0 \\ 0 & \sqrt{1-u^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.4}$$

In order to simplify our discussion, in this paper, we only suppose that u is a arbitrary function of radius r; that is,

$$u = u(r), \tag{3.5}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial \varphi} = 0. \tag{3.6}$$

It can be easily proved that the transformation matrices satisfy

$$\frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \xi^\nu} = \delta^\mu_\nu; \tag{3.7}$$

$$\frac{\partial x^\alpha}{\partial \xi^\mu}\frac{\partial \xi^\mu}{\partial x^\beta} = \delta^\alpha_\beta; \tag{3.8}$$

$$\frac{\partial^2 t}{\partial t'\partial r'} - \frac{\partial^2 t}{\partial r'\partial t'} = \frac{uu'}{1-u^2}, \tag{3.9}$$

$$\frac{\partial^2 t'}{\partial t \partial r} - \frac{\partial^2 t'}{\partial r \partial t} = -\frac{u u'}{(1-u^2)^{3/2}}, \tag{3.10}$$

where $u' = \frac{\mathrm{d}u}{\mathrm{d}r}$.

This transformation is a kind of general coordinate transformations. Now, let's study the transformation rules of some fundamental quantities and basic physical equations. Under this transformation, the metric tensors transform as:

$$\eta_{\mu\nu} \to g_{\alpha\beta} = \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial \xi^\nu}{\partial x^\beta}\eta_{\mu\nu} = \begin{pmatrix} -1+u^2 & 0 & 0 & 0 \\ 0 & \frac{1}{1-u^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}, \tag{3.11}$$

$$\eta^{\mu\nu} \to g^{\alpha\beta} = \frac{\partial x^\alpha}{\partial \xi^\mu}\frac{\partial x^\beta}{\partial \xi^\nu}\eta^{\mu\nu} = \begin{pmatrix} -\frac{1}{1-u^2} & 0 & 0 & 0 \\ 0 & 1-u^2 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix}. \tag{3.12}$$

The above transformation rules of metric tensors are completely the same as those in general relativity. But the affine connection transforms as:

$$\Gamma^\lambda_{\mu\nu} \to \Gamma^\gamma_{\alpha\beta} = \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial \xi^\nu}{\partial x^\beta}\frac{\partial x^\gamma}{\partial \xi^\lambda}\Gamma^\lambda_{\mu\nu} + \frac{\partial x^\gamma}{\partial \xi^\mu}\frac{\partial^2 \xi^\mu}{\partial x^\beta \partial x^\alpha} + B^\gamma_{\alpha\beta}. \tag{3.13}$$

If $B^\gamma_{\alpha\beta}$ vanish, the transformation rule of affine connection is completely the same as that of general relativity. But, for the present case, it does not vanish:

$$B^{t'}_{r't'} = -\frac{u u'}{(1-u^2)^{3/2}}, \tag{3.14}$$

$$B^{r'}_{t't'} = -u u' \sqrt{1-u^2}, \tag{3.15}$$

and other components of $B^\gamma_{\alpha\beta}$ vanish. So, in the present case, the transformation rule of affine connection is different from that of general relativity, which will cause serious problems to the foundations of general relativity. Because the affine connection transforms in a different way than that of general relativity, under this transformations, the curvature tensor does not transforms covariantly:

$$R^\sigma_{\mu\nu\lambda} \to \mathbb{R}^\delta_{\alpha\beta\gamma} = \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial \xi^\nu}{\partial x^\beta}\frac{\partial \xi^\lambda}{\partial x^\gamma}\frac{\partial x^\delta}{\partial \xi^\sigma}R^\sigma_{\mu\nu\lambda} + G^\delta_{\alpha\beta\gamma}, \tag{3.16}$$

where $\mathbb{R}^{\delta}_{\alpha\beta\gamma}$ is the curvature tensor in coordinate system x^{α} and $G^{\delta}_{\alpha\beta\gamma}$ is the non-homogeneous term which violates covariance of curvature tensor. For the present transformations, the non-homogeneous term $G^{\delta}_{\alpha\beta\gamma}$ can be calculated explicitly:

$$G^{t'}_{r't'r'} = -\frac{(u')^2 + uu'' - u^3u''}{(1-u^2)^3} \tag{3.17}$$

$$G^{t'}_{r'r't'} = \frac{(u')^2 + uu'' - u^3u''}{(1-u^2)^3} \tag{3.18}$$

$$G^{t'}_{\theta't'\theta'} = -\frac{ruu'}{1-u^2} \tag{3.19}$$

$$G^{t'}_{\theta'\theta't'} = \frac{ruu'}{1-u^2} \tag{3.20}$$

$$G^{t'}_{\varphi't'\varphi'} = -\frac{ruu'\sin^2\theta}{1-u^2} \tag{3.21}$$

$$G^{t'}_{\varphi'\varphi't'} = \frac{ruu'\sin^2\theta}{1-u^2} \tag{3.22}$$

$$G^{r'}_{t't'r'} = -\frac{(u')^2 + uu'' - u^3u''}{1-u^2} \tag{3.23}$$

$$G^{r'}_{t'r't'} = \frac{(u')^2 + uu'' - u^3u''}{1-u^2} \tag{3.24}$$

$$G^{\theta'}_{t't'\theta'} = -\frac{uu'}{r} \tag{3.25}$$

$$G^{\theta'}_{t'\theta't'} = \frac{uu'}{r} \tag{3.26}$$

$$G^{\varphi'}_{t't'\varphi'} = -\frac{uu'}{r} \tag{3.27}$$

$$G^{\varphi'}_{t'\varphi't'} = \frac{uu'}{r} \tag{3.28}$$

where $u'' = \frac{\mathrm{d}^2u}{\mathrm{d}r^2}$. Other component of $G^{\delta}_{\alpha\beta\gamma}$ vanish Because the curvature tensor of flat space-time vanish

$$R^{\sigma}_{\mu\nu\lambda} = 0, \tag{3.29}$$

the curvature tensor $\mathbb{R}^{\delta}_{\alpha\beta\gamma}$ of the transformed space-time equals to the non-homogeneous term $G^{\delta}_{\alpha\beta\gamma}$

$$\mathbb{R}^{\delta}_{\alpha\beta\gamma} = G^{\delta}_{\alpha\beta\gamma}. \tag{3.30}$$

Eq.(3.16) violates the transformation rules given by general relativity. According to general relativity, because curvature tensor is a 4th rank tensor and the curvature tensor of flat space-time vanish, the curvature tensor $\mathbb{R}^{\delta}_{\alpha\beta\gamma}$ of the new space-time must vanish either, which contradicts with eq.(3.30). After index contraction, we can obtain the transformation rule for Recci tensor from eq.(3.16):

$$R_{\mu\nu} \to \mathbb{R}_{\alpha\beta} = \frac{\partial\xi^{\mu}}{\partial x^{\alpha}}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}R_{\mu\nu} + G_{\alpha\beta} \tag{3.31}$$

where

$$G_{\alpha\beta} = G^{\gamma}_{\alpha\gamma\beta}. \tag{3.32}$$

The non-vanishing component of $G_{\alpha\beta}$ are

$$G_{t't'} = \frac{2uu'}{r} + \frac{(u')^2 + uu'' - u^3u''}{1-u^2}, \tag{3.33}$$

$$G_{r'r'} = -\frac{(u')^2 + uu'' - u^3u''}{(1-u^2)^3}, \tag{3.34}$$

$$G_{\theta'\theta'} = -\frac{ruu'}{1-u^2}, \tag{3.35}$$

$$G_{\varphi'\varphi'} = -\frac{ruu'\sin^2\theta}{1-u^2}. \tag{3.36}$$

Because of the existence of the non-homogeneous term $G_{\alpha\beta}$, though the Recci tensor of flat space-time vanishes, the Recci tensor of the new space-time does not vanish. The corresponding transformation rule of curvature scalar R is

$$R \to \mathbb{R} = R + G, \tag{3.37}$$

where

$$G = g^{\alpha\beta}G_{\alpha\beta}. \tag{3.38}$$

In the present case, G is

$$G = 2 \cdot \frac{-2uu' + 2u^3u' - r(u')^2 - ruu'' + ru^3u''}{r(1-u^2)^2}. \tag{3.39}$$

Eq.(3.37) means that curvature scalar R is no longer a real scalar under this transformation. In other words, curvature scalar R changes its magnitude under this transformation, or that this transformation changes the space-time curvature; that is, it can change a flat space-time into a curved space-time or change a curved space-time into a flat space-time. The transformation rules of affine connection, curvature tensor, Recci tensor and curvature scalar are different from those in general relativity.

Define covariant derivatives as

$$\nabla_\mu A_\nu = \frac{\partial A_\nu}{\partial \xi^\mu} - \Gamma^\lambda_{\nu\mu} A_\lambda, \tag{3.40}$$

$$\nabla_\mu A^\nu = \frac{\partial A^\nu}{\partial \xi^\mu} + \Gamma^\nu_{\mu\lambda} A^\lambda. \tag{3.41}$$

Under this transformation, the transformation rules of covariant derivatives are

$$\nabla_\mu A_\nu \to \nabla_\alpha A_\beta = \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial \xi^\nu}{\partial x^\beta}\nabla_\mu A_\nu - B^\gamma_{\beta\alpha}\frac{\partial \xi^\nu}{\partial x^\gamma}A_\nu, \tag{3.42}$$

$$\nabla_\mu A^\nu \to \nabla_\alpha A^\beta = \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \xi^\nu}\nabla_\mu A^\nu + B^\beta_{\alpha\gamma}\frac{\partial x^\gamma}{\partial \xi^\nu}A^\nu + \frac{\partial \xi^\mu}{\partial x^\alpha}\left(\frac{\partial^2 x^\beta}{\partial \xi^\mu \partial \xi^\nu} - \frac{\partial^2 x^\beta}{\partial \xi^\nu \partial \xi^\mu}\right)A^\nu. \tag{3.43}$$

Therefore, the covariant derivatives are not a real covariant under this transformation. It can be proved that

$$\nabla_{t'} A_{r'} = \frac{\partial \xi^\mu}{\partial t'}\frac{\partial \xi^\nu}{\partial r'}\nabla_\mu A_\nu + \frac{uu'}{1-u^2}A_t, \tag{3.44}$$

$$\nabla_{t'} A_{t'} = \frac{\partial \xi^\mu}{\partial t'}\frac{\partial \xi^\nu}{\partial t'}\nabla_\mu A_\nu + uu'A_r, \tag{3.45}$$

$$\nabla_{t'} A^{r'} = \frac{\partial \xi^\mu}{\partial t'}\frac{\partial r'}{\partial \xi^\nu}\nabla_\mu A^\nu - uu'A^t \tag{3.46}$$

$$\nabla_{t'} A^{t'} = \frac{\partial \xi^\mu}{\partial t'} \frac{\partial t'}{\partial \xi^\nu} \nabla_\mu A^\nu - \frac{uu'}{1-u^2} A^r, \tag{3.47}$$

and other components of covariant derivative transform covariantly. Therefore, under the transformation eq.(3.1), the conventional covariant derivatives defined in general relativity do not transform covariantly, which will cause serious problems to the foundations of general relativity.

4 Transformation Rules of Basic Physical Equations

Now, let's discuss the transformation rules of basic physical equations. It is generally believed that, in general relativity, all basic physical equations should transform covariantly under a general coordinate transformation. Here, we directly calculate the transformation rule of these basic physical equations eqs.(2.5 - 2.8) and eq.(2.10) to see whether they transform covariantly. Without losing generality, suppose that the basic physical equations in reference system ξ^μ are given by eqs.(2.5 - 2.8) and eq.(2.10). We will directly calculate the corresponding physical equations in the new coordinate system x^α.

Suppose that the conserved current J^μ transforms covariantly under the transformation eq.(3.1), then eq.(2.5) will be transformed into the following form

$$\nabla_\alpha J^\alpha = B^\alpha_{\alpha\gamma} \frac{\partial x^\gamma}{\partial \xi^\nu} J^\nu + \frac{\partial \xi^\mu}{\partial x^\alpha} \left(\frac{\partial^2 x^\alpha}{\partial \xi^\mu \partial \xi^\nu} - \frac{\partial^2 x^\alpha}{\partial \xi^\nu \partial \xi^\mu} \right) J^\nu. \tag{4.1}$$

It can be written out explicitly

$$\nabla_\alpha J^\alpha = - \frac{uu'}{1-u^2} J^r. \tag{4.2}$$

Because the right hand side of eq.(4.2) does not vanish, current J^α is not a conserved current in the new coordinate system x^α, or in the new coordinate system, the continuity equation does not hold. The reason is simple: the transformation matrices eqs.(3.3 - 3.4) or new coordiantes x^α carry some dynamics. The space-time itself carries some dynamics, the physical vacuum of a physical system is not a conserved system, so the physical system alone is not a conserved system. The unite system of physical vacuum and physical system is a conserved system.

Under the transformation eq.(3.1), the geodesic equation eq.(2.6) is changed into

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} - B^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial \xi^\nu}\frac{\partial x^\gamma}{\partial \xi^\lambda}\frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^\lambda}{\mathrm{d}\tau} = 0. \tag{4.3}$$

This equation can be written out explicitly

$$\frac{\mathrm{d}^2 t'}{\mathrm{d}\tau^2} + \Gamma^{t'}_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = -\frac{uu'}{(1-u^2)^{3/2}}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}, \tag{4.4}$$

$$\frac{\mathrm{d}^2 r'}{\mathrm{d}\tau^2} + \Gamma^{r'}_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = -\frac{uu'}{\sqrt{1-u^2}}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2, \tag{4.5}$$

$$\frac{\mathrm{d}^2 \theta'}{\mathrm{d}\tau^2} + \Gamma^{\theta'}_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = 0, \tag{4.6}$$

$$\frac{\mathrm{d}^2 \varphi'}{\mathrm{d}\tau^2} + \Gamma^{\varphi'}_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = 0. \tag{4.7}$$

Because the right hand sides of eqs.(4.4 – 4.5) do not vanish, the geodesic equation is not satisfied in the new coordinate system x^α. Or in other words, if the mass point moves along the geodesic line in the original reference system ξ^μ, it will not move along the geodesic line in the new coordinate system x^α. It means that, in an arbitrary curved space-time and in an arbitrary coordinate system, a mass point will not move along the geodesic line, or in most general case, the geodesic line is not the trajectory of a free mass point.

According to general relativity, the parallel transport equation in an arbitrary curved space-time is given by eq.(2.7). Supposed that in the reference system ξ^μ, the parallel transport equation is satisfied. After the transformation eq.(3.1), it will be changed into

$$\frac{\mathrm{D}A^\alpha}{\mathrm{D}\tau} = \frac{\partial x^\beta}{\partial \xi^\nu}\frac{\partial x^\gamma}{\partial \xi^\lambda}B^\alpha_{\beta\gamma}\frac{\mathrm{d}\xi^\lambda}{\mathrm{d}\tau}A^\nu. \tag{4.8}$$

Its explicit form is

$$\frac{\mathrm{D}A^{t'}}{\mathrm{D}\tau} = -\frac{uu'}{(1-u^2)^{3/2}}\frac{\mathrm{d}t}{\mathrm{d}\tau}A^r, \tag{4.9}$$

$$\frac{\mathrm{D}A^{r'}}{\mathrm{D}\tau} = -\frac{uu'}{\sqrt{1-u^2}}\frac{\mathrm{d}t}{\mathrm{d}\tau}A^t, \tag{4.10}$$

$$\frac{\mathrm{D}A^{\theta'}}{\mathrm{D}\tau} = 0, \tag{4.11}$$

$$\frac{\mathrm{D}A^{\varphi'}}{\mathrm{D}\tau} = 0. \tag{4.12}$$

For a covariant vector B_μ, according to general relativity, its parallel transport equation is

$$\frac{\mathrm{D}B_\mu}{\mathrm{D}\tau} = 0, \tag{4.13}$$

where

$$\frac{\mathrm{D}B_\mu}{\mathrm{D}\tau} = \frac{\mathrm{d}B_\mu}{\mathrm{d}\tau} - \Gamma^\lambda_{\mu\nu} \frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau} B_\lambda. \tag{4.14}$$

Suppose that in the original reference system, covariant vector B_μ satisfies parallel transport equation eq.(4.13). Then after the transformation eq.(3.1), it will become

$$\frac{\mathrm{D}B_\alpha}{\mathrm{D}\tau} = -\frac{\partial x^\beta}{\partial \xi^\nu} \frac{\partial \xi^\lambda}{\partial x^\gamma} B^\gamma_{\alpha\beta} \frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau} B_\lambda. \tag{4.15}$$

It can be written out explicitly

$$\frac{\mathrm{D}B_{t'}}{\mathrm{D}\tau} = \frac{uu'}{\sqrt{1-u^2}} \frac{\mathrm{d}t}{\mathrm{d}\tau} B_r, \tag{4.16}$$

$$\frac{\mathrm{D}B_{r'}}{\mathrm{D}\tau} = \frac{uu'}{(1-u^2)^{3/2}} \frac{\mathrm{d}t}{\mathrm{d}\tau} B_t, \tag{4.17}$$

$$\frac{\mathrm{D}B_{\theta'}}{\mathrm{D}\tau} = 0, \tag{4.18}$$

$$\frac{\mathrm{D}B_{\varphi'}}{\mathrm{D}\tau} = 0. \tag{4.19}$$

Because the right hand side of eqs.(4.9 – 4.10) and eqs.(4.16 – 4.17) do not vanish, the parallel transport equations do not hold in the new coordinate system.

In general relativity, the energy-momentum conservation equation is given by eq.(2.8). Suppose that it holds in the reference system ξ^μ. Then the transformation eq.(3.1) will make it change into

$$\nabla_\alpha T^{\beta\alpha} = \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \xi^\nu} \left(\frac{\partial^2 x^\alpha}{\partial \xi^\mu \partial \xi^\lambda} - \frac{\partial^2 x^\alpha}{\partial \xi^\lambda \partial \xi^\mu} \right) T^{\nu\lambda} + \frac{\partial x^\alpha}{\partial \xi^\lambda} \frac{\partial x^\delta}{\partial \xi^\sigma} B^\beta_{\alpha\delta} T^{\sigma\lambda} + \frac{\partial x^\beta}{\partial \xi^\nu} \frac{\partial x^\delta}{\partial \xi^\sigma} B^\alpha_{\alpha\delta} T^{\nu\sigma}. \tag{4.20}$$

Its explicit form is

$$\nabla_\alpha T^{t'\alpha} = -\frac{2uu'}{(1-u^2)^{3/2}} T^{tr}, \tag{4.21}$$

$$\nabla_\alpha T^{r'\alpha} = -\frac{uu'}{\sqrt{1-u^2}} (T^{tt} + T^{rr}), \tag{4.22}$$

$$\nabla_\alpha T^{\theta'\alpha} = -\frac{uu'}{1-u^2} T^{\theta r}, \tag{4.23}$$

$$\nabla_\alpha T^{\varphi'\alpha} = -\frac{uu'}{1-u^2} T^{\varphi r}. \tag{4.24}$$

These equations mean that, in the new coordinate system x^α, energy-momentum is no longer conserved. This result is a little surprising, but is easily to understand. Suppose that the reference system ξ^μ is a local inertial reference system and a mass point moves freely in it. Because of energy-momentum conservation, the speed of the mass point will not change with time. But in the new coordinate system x^α, because of its variable motion, the speed of the mass point will change with time. So, the observer in the new coordinate system x^α will find that the energy-momentum of the mass point changes with time, or in other words, its energy-momentum is not conserved.

Finally, let's discuss Einstein field equation eq.(2.10). We also suppose that it holds in the original reference ssytem ξ^μ. Transformation eq.(3.1) changes it into

$$\mathbb{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\mathbb{R} + 8\pi G_N T_{\alpha\beta} = G_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} G. \tag{4.25}$$

If the right hand side of the above equation vanish, the Einstein field equation still holds in the new coordinate system x^α. However, it does not vanish. In the new coordinate system x^α, the diagonal component equations are

$$\mathbb{R}_{t't'} - \frac{1}{2} g_{t't'}\mathbb{R} + 8\pi G_N T_{t't'} = 0, \tag{4.26}$$

$$\mathbb{R}_{r'r'} - \frac{1}{2} g_{r'r'}\mathbb{R} + 8\pi G_N T_{r'r'} = \frac{2uu'}{r(1-u^2)^2}, \tag{4.27}$$

$$\mathbb{R}_{\theta'\theta'} - \frac{1}{2} g_{\theta'\theta'}\mathbb{R} + 8\pi G_N T_{\theta'\theta'} = \frac{ruu' - ru^3u' + r^2(u')^2 + r^2uu'' - r^2u^3u''}{(1-u^2)^2}, \tag{4.28}$$

$$\mathbb{R}_{\varphi'\varphi'} - \frac{1}{2} g_{\varphi'\varphi'} \mathbb{R} + 8\pi G_N T_{\varphi'\varphi'} = \frac{ruu' - ru^3 u' + r^2 (u')^2 + r^2 u u'' - r^2 u^3 u''}{(1-u^2)^2} \sin^2\theta. \quad (4.29)$$

Therefore, it is clear that Einstein field equation does not transform covriantly under a very simple coordinate transformation eq.(3.1). This result is obtained by direct calculation. Einstein field equation is not general covariant. It means that, if in the original reference system ξ^μ, the Einstein field equation is satisfied, it will not be satisfied in the new coordinate system x^α. So, if an observer in an arbitrary curved space-time, he will not know that that whether the Einstein field equation is satisfied or not in his coordinate system, or in other words, Einstein field equation does not hold in an arbitrary coordinate system of an arbitrary curved space-time.

5 A General theory on General Coordinate Transformation

In the chapter 4, through a simple example, we know that basic physical equations are not covariant under a general coordinate transformation, because almost all physical equations change their forms under the general coordinate transformation eq.(3.1). But in general relativity, it is also proved that all physical equations preserve their forms under a general coordinate transformation. These two results are contradicted each other. In this chapter, we will develop a general theory on general coordinate transformations, and study what cause the violation of general covariance in general relativity.

First, we need to point out that transformation eq.(3.1) is a kind of general coordinate transformations. It is a simple example of general coordinate transformations. Essentially speaking, local Lorentz transformation belongs to the same kind of general coordinate transformations[26]. For a general coordinate transformation,

$$\mathrm{d}\xi^\mu \to \mathrm{d}x^\alpha = \frac{\partial x^\alpha}{\partial \xi^\mu} \mathrm{d}\xi^\mu, \quad (5.1)$$

$$\frac{\partial}{\partial \xi^\mu} \to \frac{\partial}{\partial x^\alpha} = \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial}{\partial \xi^\mu}, \quad (5.2)$$

the transformation matrix must satisfy the following restriction:

$$\frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \xi^\nu} = \delta^\mu_\nu; \quad (5.3)$$

$$\frac{\partial x^\alpha}{\partial \xi^\mu}\frac{\partial \xi^\mu}{\partial x^\beta} = \delta^\alpha_\beta. \tag{5.4}$$

There are two kinds of general coordinate transformations. The first kind is a trivial general coordinate transformation, or call it curvilinear coordinate transformation. Under this transformation, the space-time itself undergoes no changes. The only change is that we transform one curvilinear coordinate system into another curvilinear coordinate system. For curvilinear coordinate transformation, the transformation matrices satisfy:

$$\frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 \xi^\mu}{\partial x^\beta \partial x^\alpha} = 0, \tag{5.5}$$

$$\frac{\partial^2 x^\alpha}{\partial \xi^\mu \partial \xi^\nu} - \frac{\partial^2 x^\alpha}{\partial \xi^\nu \partial \xi^\mu} = 0. \tag{5.6}$$

The transformation between Cartesian coordinate system and spherical coordinate system belongs to this kind. Both transformation eq.(3.1) and local Lorentz transformation do not satisfy the above two restrictions eqs.(5.5 – 5.6), so they do not belong to this kind.

For the second kind of general coordinate transformations, the transformation matrix do not satisfy the above two restrictions eqs.(5.5 – 5.6), i.e.

$$\frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 \xi^\mu}{\partial x^\beta \partial x^\alpha} \neq 0, \tag{5.7}$$

$$\frac{\partial^2 x^\alpha}{\partial \xi^\mu \partial \xi^\nu} - \frac{\partial^2 x^\alpha}{\partial \xi^\nu \partial \xi^\mu} \neq 0. \tag{5.8}$$

A lot of important general coordinate transformations, such as transformation eq.(3.1) and local Lorentz transformation, belong to this kind of general coordinate transformations. On of the most important transformation in general relativity, the transformation from local inertial reference system to an arbitrary curved space-time, belongs to this kind. In this chapter, we mainly discuss this kind of general coordinate transformations. Later we will know that the violation of eqs(5.5 – 5.6) causes the violation of general covariance in general relativity.

The general coordinate system before the transformation is denoted as ξ^μ (please note that in this chapter, ξ^μ is no longer coordinate system of flat space-time.) and that after the transformation is denoted as x^α. The transformation from one general coordinate system ξ^μ to another general coordinate system x^α is

$$\mathrm{d}\xi^\mu \to \mathrm{d}x^\alpha = \frac{\partial x^\alpha}{\partial \xi^\mu}\mathrm{d}\xi^\mu, \tag{5.9}$$

$$\frac{\partial}{\partial \xi^\mu} \to \frac{\partial}{\partial x^\alpha} = \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial}{\partial \xi^\mu}, \tag{5.10}$$

where transformation matrices $\frac{\partial x^\alpha}{\partial \xi^\mu}$ and $\frac{\partial \xi^\mu}{\partial x^\alpha}$ satisfy eqs.(3.7 – 3.8) and eqs(5.7 – 5.8). Under this transformation, the metric tensors transform as:

$$\eta_{\mu\nu} \to g_{\alpha\beta} = \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial \xi^\nu}{\partial x^\beta}\eta_{\mu\nu}, \tag{5.11}$$

$$\eta^{\mu\nu} \to g^{\alpha\beta} = \frac{\partial x^\alpha}{\partial \xi^\mu}\frac{\partial x^\beta}{\partial \xi^\nu}\eta^{\mu\nu}, \tag{5.12}$$

where $\eta^{\mu\nu}$ is the metric tensor of the coordinate system ξ^μ(In this chapter, $\eta^{\mu\nu}$ is not Minkowski metric.). The affine connection transforms as:

$$\Gamma^\lambda_{\mu\nu} \to \Gamma^\gamma_{\alpha\beta} = \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial \xi^\nu}{\partial x^\beta}\frac{\partial x^\gamma}{\partial \xi^\lambda}\Gamma^\lambda_{\mu\nu} + \frac{\partial x^\gamma}{\partial \xi^\mu}\frac{\partial^2 \xi^\mu}{\partial x^\beta \partial x^\alpha} + B^\gamma_{\alpha\beta}, \tag{5.13}$$

where

$$\begin{aligned} B^\gamma_{\alpha\beta} = & \tfrac{1}{2}\tfrac{\partial x^\gamma}{\partial \xi^\mu}\left(\tfrac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\beta} - \tfrac{\partial^2 \xi^\mu}{\partial x^\beta \partial x^\alpha}\right) \\ & + \tfrac{1}{2}\tfrac{\partial \xi^\mu}{\partial x^\alpha}\tfrac{\partial x^\gamma}{\partial \xi^\lambda}\tfrac{\partial x^\delta}{\partial \xi^{\sigma_1}}\eta_{\mu\nu}\eta^{\lambda\sigma_1}\left(\tfrac{\partial^2 \xi^\nu}{\partial x^\beta \partial x^\delta} - \tfrac{\partial^2 \xi^\nu}{\partial x^\delta \partial x^\beta}\right) \\ & + \tfrac{1}{2}\tfrac{\partial \xi^\nu}{\partial x^\beta}\tfrac{\partial x^\gamma}{\partial \xi^\lambda}\tfrac{\partial x^\delta}{\partial \xi^{\sigma_1}}\eta_{\mu\nu}\eta^{\lambda\sigma_1}\left(\tfrac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\delta} - \tfrac{\partial^2 \xi^\mu}{\partial x^\delta \partial x^\alpha}\right). \end{aligned} \tag{5.14}$$

For the first kind of general coordinate transformations, eqs.(5.5 – 5.6) are satisfied, so $B^\gamma_{\alpha\beta}$ vanishes and the transformation rule of affine connection becomes the same as that of general relativity. For the second kind of general coordinate transformations, $B^\gamma_{\alpha\beta}$ does not vanish and the transformation rule of affine connection is different that of general relativity. The appearance of the non-homogeneous term $B^\gamma_{\alpha\beta}$ is the origin of the violation of general covariance of general relativity, and the violation of eqs.(5.5 – 5.6) is the origin of the appearance of $B^\gamma_{\alpha\beta}$, so the violation of eqs.(5.5 – 5.6) is the essential origin of the violation of general covariance of general relativity.

Under the second kind of general coordinate transformations, curvature tensor transforms as

$$R^{\sigma}_{\mu\nu\lambda} \to \mathbb{R}^{\delta}_{\alpha\beta\gamma} = \frac{\partial\xi^{\mu}}{\partial x^{\alpha}}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}\frac{\partial\xi^{\lambda}}{\partial x^{\gamma}}\frac{\partial x^{\delta}}{\partial\xi^{\sigma}} R^{\sigma}_{\mu\nu\lambda} + G^{\delta}_{\alpha\beta\gamma}, \tag{5.15}$$

where $\mathbb{R}^{\delta}_{\alpha\beta\gamma}$ is the curvature tensor in coordinate system x^{α} and $G^{\delta}_{\alpha\beta\gamma}$ is the non-homogeneous term which violates general covariance of curvature tensor. Explicit form of $G^{\delta}_{\alpha\beta\gamma}$ is quite complicated

$$G^{\delta}_{\alpha\beta\gamma} = \sum_{i=1}^{8} G^{\delta}_{(i)\alpha\beta\gamma}, \tag{5.16}$$

where

$$\begin{aligned} G^{\delta}_{(1)\alpha\beta\gamma} &= \tfrac{1}{2}\tfrac{\partial\xi^{\nu}}{\partial x^{\beta}}\tfrac{\partial x^{\delta}}{\partial\xi^{\kappa}}\left(\tfrac{\partial^2\xi^{\mu}}{\partial x^{\gamma}\partial x^{\alpha}} - \tfrac{\partial^2\xi^{\mu}}{\partial x^{\alpha}\partial x^{\gamma}}\right)\Gamma^{\kappa}_{\mu\nu} - (\beta \leftrightarrow \gamma) \\ &+ \tfrac{\partial\xi^{\mu}}{\partial x^{\alpha}}\tfrac{\partial x^{\delta}}{\partial\xi^{\kappa}}\left(\tfrac{\partial^2\xi^{\nu}}{\partial x^{\gamma}\partial x^{\beta}} - \tfrac{\partial^2\xi^{\nu}}{\partial x^{\beta}\partial x^{\gamma}}\right)\Gamma^{\kappa}_{\mu\nu}, \end{aligned} \tag{5.17}$$

$$\begin{aligned} G^{\delta}_{(2)\alpha\beta\gamma} &= \tfrac{\partial\xi^{\mu}}{\partial x^{\alpha}}\tfrac{\partial\xi^{\nu}}{\partial x^{\beta}}\tfrac{\partial\xi^{\lambda}}{\partial x^{\gamma}}\left(\tfrac{\partial^2 x^{\delta}}{\partial\xi^{\lambda}\partial\xi^{\kappa}} - \tfrac{\partial^2 x^{\delta}}{\partial\xi^{\kappa}\partial\xi^{\lambda}}\right)\Gamma^{\kappa}_{\mu\nu} - (\beta \leftrightarrow \gamma) \\ &+ \tfrac{1}{2}\tfrac{\partial\xi^{\mu}}{\partial x^{\alpha}}\tfrac{\partial\xi^{\nu}}{\partial x^{\beta}}\tfrac{\partial x^{\alpha_1}}{\partial\xi^{\mu_1}}\tfrac{\partial x^{\delta}}{\partial\xi^{\kappa}}\left(\tfrac{\partial^2\xi^{\kappa}}{\partial x^{\gamma}\partial x^{\alpha_1}} - \tfrac{\partial^2\xi^{\kappa}}{\partial x^{\alpha_1}\partial x^{\gamma}}\right)\Gamma^{\mu_1}_{\mu\nu} - (\beta \leftrightarrow \gamma), \end{aligned} \tag{5.18}$$

$$\begin{aligned} G^{\delta}_{(3)\alpha\beta\gamma} &= \tfrac{1}{2}\tfrac{\partial\xi^{\mu}}{\partial x^{\alpha}}\tfrac{\partial\xi^{\nu}}{\partial x^{\beta}}\tfrac{\partial\xi^{\lambda}}{\partial x^{\gamma}}\tfrac{\partial x^{\alpha_1}}{\partial\xi^{\mu_1}} g^{\delta\varepsilon}\eta_{\lambda\mu_2}\left(\tfrac{\partial^2\xi^{\mu_2}}{\partial x^{\alpha_1}\partial x^{\varepsilon}} - \tfrac{\partial^2\xi^{\mu_2}}{\partial x^{\varepsilon}\partial x^{\alpha_1}}\right)\Gamma^{\mu_1}_{\mu\nu} - (\beta \leftrightarrow \gamma) \\ &+ \tfrac{1}{2}\tfrac{\partial\xi^{\mu}}{\partial x^{\alpha}}\tfrac{\partial\xi^{\nu}}{\partial x^{\beta}} g^{\delta\varepsilon}\eta_{\lambda\mu_1}\left(\tfrac{\partial^2\xi^{\lambda}}{\partial x^{\gamma}\partial x^{\varepsilon}} - \tfrac{\partial^2\xi^{\lambda}}{\partial x^{\varepsilon}\partial x^{\gamma}}\right)\Gamma^{\mu_1}_{\mu\nu} - (\beta \leftrightarrow \gamma) \\ &+ \tfrac{1}{2}\tfrac{\partial\xi^{\mu}}{\partial x^{\alpha}}\tfrac{\partial\xi^{\mu_1}}{\partial x^{\alpha_1}}\tfrac{\partial\xi^{\lambda}}{\partial x^{\gamma}}\tfrac{\partial x^{\delta}}{\partial\xi^{\kappa}} g^{\alpha_1\varepsilon}\eta_{\mu\nu}\left(\tfrac{\partial^2\xi^{\nu}}{\partial x^{\beta}\partial x^{\varepsilon}} - \tfrac{\partial^2\xi^{\nu}}{\partial x^{\varepsilon}\partial x^{\beta}}\right)\Gamma^{\kappa}_{\lambda\mu_1} - (\beta \leftrightarrow \gamma) \\ &+ \tfrac{1}{2}\tfrac{\partial\xi^{\nu}}{\partial x^{\beta}}\tfrac{\partial\xi^{\mu_1}}{\partial x^{\alpha_1}}\tfrac{\partial\xi^{\lambda}}{\partial x^{\gamma}}\tfrac{\partial x^{\delta}}{\partial\xi^{\kappa}} g^{\alpha_1\varepsilon}\eta_{\mu\nu}\left(\tfrac{\partial^2\xi^{\mu}}{\partial x^{\alpha}\partial x^{\varepsilon}} - \tfrac{\partial^2\xi^{\mu}}{\partial x^{\varepsilon}\partial x^{\alpha}}\right)\Gamma^{\kappa}_{\lambda\mu_1} - (\beta \leftrightarrow \gamma), \end{aligned} \tag{5.19}$$

$$
\begin{aligned}
G^{\delta}_{(4)\alpha\beta\gamma} &= \tfrac{1}{2}\frac{\partial\xi^{\nu}}{\partial x^{\beta}} g^{\delta\varepsilon}\eta_{\mu\nu}\frac{\partial}{\partial x^{\gamma}}\left(\frac{\partial^2\xi^{\mu}}{\partial x^{\alpha}\partial x^{\varepsilon}} - \frac{\partial^2\xi^{\mu}}{\partial x^{\varepsilon}\partial x^{\alpha}}\right) - (\beta\leftrightarrow\gamma) \\
&+\tfrac{1}{2}\frac{\partial\xi^{\mu}}{\partial x^{\alpha}} g^{\delta\varepsilon}\eta_{\mu\nu}\frac{\partial}{\partial x^{\gamma}}\left(\frac{\partial^2\xi^{\nu}}{\partial x^{\beta}\partial x^{\varepsilon}} - \frac{\partial^2\xi^{\nu}}{\partial x^{\varepsilon}\partial x^{\beta}}\right) - (\beta\leftrightarrow\gamma) \qquad (5.20) \\
&+\tfrac{1}{2}\frac{\partial x^{\delta}}{\partial\xi^{\kappa}}\left(\frac{\partial^3\xi^{\kappa}}{\partial x^{\gamma}\partial x^{\alpha}\partial x^{\beta}} + \frac{\partial^3\xi^{\kappa}}{\partial x^{\gamma}\partial x^{\beta}\partial x^{\alpha}} - \frac{\partial^3\xi^{\kappa}}{\partial x^{\beta}\partial x^{\alpha}\partial x^{\gamma}} - \frac{\partial^3\xi^{\kappa}}{\partial x^{\beta}\partial x^{\gamma}\partial x^{\alpha}}\right),
\end{aligned}
$$

$$
\begin{aligned}
G^{\delta}_{(5)\alpha\beta\gamma} &= \tfrac{1}{2}\frac{\partial\xi^{\lambda}}{\partial x^{\gamma}}\left(\frac{\partial^2 x^{\delta}}{\partial\xi^{\lambda}\partial\xi^{\kappa}} - \frac{\partial^2 x^{\delta}}{\partial\xi^{\kappa}\partial\xi^{\lambda}}\right)\left(\frac{\partial^2\xi^{\kappa}}{\partial x^{\alpha}\partial x^{\beta}} + \frac{\partial^2\xi^{\kappa}}{\partial x^{\beta}\partial x^{\alpha}}\right) - (\beta\leftrightarrow\gamma) \\
&+\tfrac{1}{4}\frac{\partial x^{\alpha_1}}{\partial\xi^{\mu_1}}\frac{\partial x^{\delta}}{\partial\xi^{\kappa}}\left(\frac{\partial^2\xi^{\kappa}}{\partial x^{\gamma}\partial x^{\alpha_1}} - \frac{\partial^2\xi^{\kappa}}{\partial x^{\alpha_1}\partial x^{\gamma}}\right)\left(\frac{\partial^2\xi^{\mu_1}}{\partial x^{\alpha}\partial x^{\beta}} + \frac{\partial^2\xi^{\mu_1}}{\partial x^{\beta}\partial x^{\alpha}}\right) - (\beta\leftrightarrow\gamma), \qquad (5.21)
\end{aligned}
$$

$$
\begin{aligned}
G^{\delta}_{(6)\alpha\beta\gamma} &= \tfrac{1}{2}\left[\frac{\partial}{\partial x^{\gamma}}\left(\frac{\partial\xi^{\mu}}{\partial x^{\alpha}} g^{\delta\varepsilon}\eta_{\mu\nu}\right)\right]\left(\frac{\partial^2\xi^{\nu}}{\partial x^{\beta}\partial x^{\varepsilon}} - \frac{\partial^2\xi^{\nu}}{\partial x^{\varepsilon}\partial x^{\beta}}\right) - (\beta\leftrightarrow\gamma) \\
&+\tfrac{1}{2}\left[\frac{\partial}{\partial x^{\gamma}}\left(\frac{\partial\xi^{\nu}}{\partial x^{\beta}} g^{\delta\varepsilon}\eta_{\mu\nu}\right)\right]\left(\frac{\partial^2\xi^{\mu}}{\partial x^{\alpha}\partial x^{\varepsilon}} - \frac{\partial^2\xi^{\mu}}{\partial x^{\varepsilon}\partial x^{\alpha}}\right) - (\beta\leftrightarrow\gamma), \qquad (5.22)
\end{aligned}
$$

$$
\begin{aligned}
G^{\delta}_{(7)\alpha\beta\gamma} &= \tfrac{1}{4}\frac{\partial x^{\alpha_1}}{\partial\xi^{\mu_1}}\frac{\partial\xi^{\lambda}}{\partial x^{\gamma}} g^{\delta\varepsilon}\eta_{\lambda\mu_2}\left(\frac{\partial^2\xi^{\mu_2}}{\partial x^{\alpha_1}\partial x^{\varepsilon}} - \frac{\partial^2\xi^{\mu_2}}{\partial x^{\varepsilon}\partial x^{\alpha_1}}\right)\left(\frac{\partial^2\xi^{\mu_1}}{\partial x^{\alpha}\partial x^{\beta}} + \frac{\partial^2\xi^{\mu_1}}{\partial x^{\beta}\partial x^{\alpha}}\right) - (\beta\leftrightarrow\gamma) \\
&+\tfrac{1}{4}\frac{\partial x^{\delta}}{\partial\xi^{\kappa}}\frac{\partial\xi^{\mu}}{\partial x^{\alpha}} g^{\alpha_1\varepsilon}\eta_{\mu\nu}\left(\frac{\partial^2\xi^{\nu}}{\partial x^{\beta}\partial x^{\varepsilon}} - \frac{\partial^2\xi^{\nu}}{\partial x^{\varepsilon}\partial x^{\beta}}\right)\left(\frac{\partial^2\xi^{\kappa}}{\partial x^{\gamma}\partial x^{\alpha_1}} + \frac{\partial^2\xi^{\kappa}}{\partial x^{\alpha_1}\partial x^{\gamma}}\right) - (\beta\leftrightarrow\gamma) \\
&+\tfrac{1}{4}\frac{\partial x^{\delta}}{\partial\xi^{\kappa}}\frac{\partial\xi^{\nu}}{\partial x^{\beta}} g^{\alpha_1\varepsilon}\eta_{\mu\nu}\left(\frac{\partial^2\xi^{\mu}}{\partial x^{\alpha}\partial x^{\varepsilon}} - \frac{\partial^2\xi^{\mu}}{\partial x^{\varepsilon}\partial x^{\alpha}}\right)\left(\frac{\partial^2\xi^{\kappa}}{\partial x^{\gamma}\partial x^{\alpha_1}} + \frac{\partial^2\xi^{\kappa}}{\partial x^{\alpha_1}\partial x^{\gamma}}\right) - (\beta\leftrightarrow\gamma) \\
&+\tfrac{1}{4} g^{\delta\varepsilon}\eta_{\lambda\mu_1}\left(\frac{\partial^2\xi^{\lambda}}{\partial x^{\gamma}\partial x^{\varepsilon}} - \frac{\partial^2\xi^{\lambda}}{\partial x^{\varepsilon}\partial x^{\gamma}}\right)\left(\frac{\partial^2\xi^{\mu_1}}{\partial x^{\alpha}\partial x^{\beta}} + \frac{\partial^2\xi^{\mu_1}}{\partial x^{\beta}\partial x^{\alpha}}\right) - (\beta\leftrightarrow\gamma), \qquad (5.23)
\end{aligned}
$$

$$
\begin{aligned}
G^{\delta}_{(8)\alpha\beta\gamma} \;=\;& \frac{1}{4}\frac{\partial\xi^{\mu}}{\partial x^{\alpha}}\frac{\partial\xi^{\lambda}}{\partial x^{\gamma}} g^{\alpha_1\varepsilon} g^{\delta\varepsilon_1}\eta_{\mu\nu}\eta_{\lambda\mu_1}\left(\frac{\partial^2\xi^{\nu}}{\partial x^{\beta}\partial x^{\varepsilon}}-\frac{\partial^2\xi^{\nu}}{\partial x^{\varepsilon}\partial x^{\beta}}\right)\left(\frac{\partial^2\xi^{\mu_1}}{\partial x^{\alpha_1}\partial x^{\varepsilon_1}}-\frac{\partial^2\xi^{\mu_1}}{\partial x^{\varepsilon_1}\partial x^{\alpha_1}}\right)-(\beta\leftrightarrow\gamma)\\
&+\frac{1}{4}\frac{\partial\xi^{\mu}}{\partial x^{\alpha}}\frac{\partial\xi^{\mu_1}}{\partial x^{\alpha_1}} g^{\alpha_1\varepsilon_1} g^{\delta\varepsilon}\eta_{\mu\nu}\eta_{\lambda\mu_1}\left(\frac{\partial^2\xi^{\nu}}{\partial x^{\beta}\partial x^{\varepsilon_1}}-\frac{\partial^2\xi^{\nu}}{\partial x^{\varepsilon_1}\partial x^{\beta}}\right)\left(\frac{\partial^2\xi^{\lambda}}{\partial x^{\gamma}\partial x^{\varepsilon}}-\frac{\partial^2\xi^{\lambda}}{\partial x^{\varepsilon}\partial x^{\gamma}}\right)-(\beta\leftrightarrow\gamma)\\
&+\frac{1}{4}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}\frac{\partial\xi^{\lambda}}{\partial x^{\gamma}} g^{\alpha_1\varepsilon} g^{\delta\varepsilon_1}\eta_{\mu\nu}\eta_{\lambda\mu_1}\left(\frac{\partial^2\xi^{\mu}}{\partial x^{\alpha}\partial x^{\varepsilon}}-\frac{\partial^2\xi^{\mu}}{\partial x^{\varepsilon}\partial x^{\alpha}}\right)\left(\frac{\partial^2\xi^{\mu_1}}{\partial x^{\alpha_1}\partial x^{\varepsilon_1}}-\frac{\partial^2\xi^{\mu_1}}{\partial x^{\varepsilon_1}\partial x^{\alpha_1}}\right)-(\beta\leftrightarrow\gamma)\\
&+\frac{1}{4}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}\frac{\partial\xi^{\mu_1}}{\partial x^{\alpha_1}} g^{\alpha_1\varepsilon} g^{\delta\varepsilon_1}\eta_{\mu\nu}\eta_{\lambda\mu_1}\left(\frac{\partial^2\xi^{\mu}}{\partial x^{\alpha}\partial x^{\varepsilon}}-\frac{\partial^2\xi^{\mu}}{\partial x^{\varepsilon}\partial x^{\alpha}}\right)\left(\frac{\partial^2\xi^{\lambda}}{\partial x^{\gamma}\partial x^{\varepsilon_1}}-\frac{\partial^2\xi^{\lambda}}{\partial x^{\varepsilon_1}\partial x^{\gamma}}\right)-(\beta\leftrightarrow\gamma).
\end{aligned}
\tag{5.24}
$$

In the above relations, symbol $(\beta \leftrightarrow \gamma)$ represents the term with exchange two index β and γ of the preceding term. From above expressions of $G^{\delta}_{\alpha\beta\gamma}$, we find that if eqs.(5.5 – 5.6) hold, $G^{\delta}_{\alpha\beta\gamma}$ strictly vanishes and the transformation of curvature tensor is covariant; i.e., the curvature tensor transforms covariantly under the first kind of general coordinate transformations. But under the second kind of general coordinate transformations, the curvature tensor does not transform covariantly. It means that, under the second kind of general coordinate transformations, curvature tensor is not a real tensor. Correspondingly, the transformation rule of Recci tensor is

$$
R_{\mu\nu} \to \mathbb{R}_{\alpha\beta} = \frac{\partial\xi^{\mu}}{\partial x^{\alpha}}\frac{\partial\xi^{\nu}}{\partial x^{\beta}} R_{\mu\nu} + G_{\alpha\beta}, \tag{5.25}
$$

where

$$
G_{\alpha\beta} = G^{\gamma}_{\alpha\gamma\beta}. \tag{5.26}
$$

Similarly, Recci tensor is not a real tensor under the second kind of general coordinate transformations. The transformation of curvature scalar R is

$$
R \to \mathbb{R} = R + G, \tag{5.27}
$$

where

$$
G = g^{\alpha\beta} G_{\alpha\beta}. \tag{5.28}
$$

Under the second kind of general coordinate transformations, G does not vanish, so the curvature scalar R change its magnitude, which means that the space-time structure was changed. The second kind of general coordinate transformations can transform a flat space-time into a curved space-time or vice versa. So, a transformation which changes curvature scalar must belong to the second kind of general coordinate transformations, for example, the transformation between local inertial reference system and a curved space-time is a second kind of general coordinate transformations.

Covariant derivatives are defined by eqs.(3.40 – 3.41). Under transformation eq.(5.1), the transformation rules for covariant derivatives are given by eqs.(3.42 – 3.43). For the second kind of general coordinate transformations, these covariant derivatives are not real covariant. Under transformation eq.(5.1), the transformation rule for continuity equation is given by eq.(4.1), that for geodesic equation is given by eq.(4.3), that for parallel transport equation is given by eq.(4.8), that for energy-momentum conservation equation is given by eq.(4.20) and that for Einstein field equation is given by eq.(4.25). We could see that all these basic physical equations can not preserve their forms under the second kind of general coordinate transformations. The violation of general covariance originates from the violation of eqs.(5.5 – 5.6), which cause $B^{\gamma}_{\alpha\beta}$ and $G^{\delta}_{\alpha\beta\gamma}$ not vanish.

6 Problems on the Foundations of General Relativity

The violation of general covariance causes serious problems to the foundations of general relativity. It is known that the *a priori* foundations of general relativity are the equivalence principle and the principle of general covariance, and these principles are directly related to the property of general covariance under general coordinate transformations. In general relativity, these two principles tell us how to obtain the laws of nature in an arbitrary curved space-time. The equivalence principle tells us that, in any point of an arbitrary curved space-time, there must exist a local inertial coordinate system and the laws of nature in it take the same form as that in unaccelerated Cartesian coordinate system in absence of gravity. Then the principle of general covariance tells us that the laws of nature in an arbitrary curved space-time are obtained through a general coordinate transformation from the local inertial coordinate system to the curved space-time. But the coordinate transformation from a local inertial coordinate system to any curved space-time belongs to the second kind of general coordinate transformations. Under the second kind of general coordinate transformations, all basic physical equations are not covariant. All physical equations given by this coordinate transformation are different from those in general relativity. So, there is a fundamental problem: what are the *a priori* foundations of general relativity? All basic physical equations in general relativity can not be obtained based on the equivalence principle and the principle of general covariance, for equations deduced based on these two principles are different from those in general relativity.

According to the theory of special relativity, the laws of nature in an inertial reference system are exactly known. If the equivalence principle strictly holds, the laws of nature in a local inertial coordinate system are also exactly known. If the nature of gravity is space-time geometry, the laws of nature in different kinds of curved space-time must be related to each other through a general coordinate transformation from one curved space-time to another curved space-time. Therefore, the laws of nature in a curved space-time can be obtained through a coordinate transformation from a local inertial coordinate system to the curved space-time. If this procedure is true, we will encounter two essential problems: (1) the basic physical equations obtained in this way are different from those in general relativity; (2) the basic physical equations can not be expressed in terms of physical observable, metric tensor and their space-time derivatives. In other words, if basic physical equations are determined in this way, an observer in a curved space-time can not determined the exact forms of these basic physical equations, for he has no information on the transformation matrix $\frac{\partial \xi^{\mu}}{\partial x^{\alpha}}$ and $\frac{\partial x^{\alpha}}{\partial \xi^{\mu}}$. Furthermore, the transformation matrix from a local inertial coordinate system to a curved space-time is not unique[26], all physical equations in the new coordinate system are also not unique, which is another serious problem on geometry picture of gravity.

Because basic physical equations are not covariant under a general coordinate transformation, what is the symmetry of gravity in general relativity?

In a word, because of the violation of general covariance under the second kind of general coordinate transformations, both the equivalence principle and the principle of general covariance can not be regarded as *a priori* foundations of general relativity.

7 Discussions

In this paper, violation of general covariance under the second kind of general coordinate transformations is studied and problems on foundations of general relativity is discussed. If gravity is space-time geometry, basic physical equations in curved space-time of different kind should be related each other through a general coordinate transformation, for these curved space-time can be related each other through a general coordinate transformation. But the basic physical equations are not covariant under a general coordinate transformation, all basic physical equations obtained from the equivalence principle and the principle of general covariance are different from those in general relativity. Besides, the forms of basic physical equations can not be uniquely determined through general coordinate transformations, for the transformation matrix from a local inertial coordinate system to a curved space-time is not unique. So, both the equivalence principle and the principle of general covariance are not *a priori* foundations of general relativity. The violation of general covariance causes serious problems to the foundations of general relativity.

Another essential problem related to general relativity is that quantum general relativity is not perturbatively renormalizable. At present, we can not set up a self-consistent quantum theory of general relativity.

Because of the great achievement of QCD, QED and unified electroweak theory in particle physics, it is generally believed that the common nature of all fundamental interactions in nature is gauge theory. Based on this belief, quantum gauge theory of gravity is proposed which is renormalizable in 4-dimensional Minkowski space-time[18, 19, 20, 21]. Quantum gauge theory of gravity is set up in the physics picture of gravity and gravity is treated as a kind of physical interactions, not space-time geometry, so it has a clear physical picture. In quantum gauge theory of gravity, four different kind of fundamental interactions in nature can be unified in a simple and beautiful way[22, 23, 24]. All these are advantage of quantum gauge theory of gravity.

Furthermore, quantum gauge theory of gravity does not have the above fundamental problems. In quantum gauge theory of gravity, gravity is treated as a kind of fundamental interactions, not space-time geometry. Physics is formulated in flat 4-dimensional Minkowski space-time, not in arbitrary curved space-time. So, in quantum gauge theory of gravity, we do not encounter the problems caused by the second kind of general coordinate transformations. Basic physical equations in quantum gauge theory of gravity are obtained through gauge principle and action principle, not through a general coordinate transformation. This is another advantage of quantum gauge theory of gravity.

Different gravity theory has different kind of symmetry, such as general relativity has the symmetry of general covariance, canonical quantum gravity has local Lorentz symmetry and quantum gauge theory of gravity has gravitational gauge symmetry. So, what is the true symmetry of gravitational interactions? According to discussions in this paper and literature [26], basic physical equations can not preserve their forms under either general coordinate transformations or local Lorentz transformations. So, the symmetry of gravitational interactions are not the symmetry of general covariance, nor the local

Lorentz symmetry. Therefore, the correct symmetry of gravitational interactions should be gravitational gauge symmetry. On the other hand, gravitational gauge symmetry is a natural requirement of gauge principle[18, 19, 20, 21].

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