General Relativity

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Introduction

These notes are based on the course "General Relativity" given by Dr. P. D. D'Eath in Cambridge in the Lent Term 1998. These typeset notes are totally unconnected with Dr. D'Eath. The recommended books for this course are discussed in the bibliography.

Other sets of notes are available for different courses. At the time of typing these courses were:

Probability	Discrete Mathematics
Analysis	Further Analysis
Methods	Quantum Mechanics
Fluid Dynamics 1	Quadratic Mathematics
Geometry	Dynamics of D.E.'s
Foundations of QM	Electrodynamics
Methods of Math. Phys	Fluid Dynamics 2
Waves (etc.)	Statistical Physics
General Relativity	Dynamical Systems

They may be downloaded from

`http://home.arachsys.com/~pdm/` or
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Chapter 1

Outline of the theory

1.1 Curved spaces

Consider a two dimensional curved surface in Euclidean $\mathbb{R}^3$, for instance with the defining equation $z = z(x, y)$. We distinguish between the extrinsic and intrinsic properties of such a surface.

The extrinsic properties describe the relation between the surface and the surrounding 3 dimensional space, for instance the extrinsic curvature if $z = z(x, y)$ is not a plane.

The intrinsic properties refer to quantities such as distance, angle and area measured within the surface. For instance, the Euclidean metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ gives the distance between nearby points in the surface as

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \left(\frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y\right)^2.$$

The surface has an intrinsic Riemannian (positive definite) metric of the form

$$\mathrm{d}s^2 = A(x, y)\,\mathrm{d}x^2 + 2B(x, y)\,\mathrm{d}x\mathrm{d}y + C(x, y)\,\mathrm{d}y^2.$$

The metric at a point P on the surface describes the geometry (distances, angles, etc.) on the plane tangent to the surface at P.

We have the freedom to change co-ordinates. If $x, y \mapsto x'(x, y), y'(x, y)$ then the metric becomes

$$\mathrm{d}s^2 = A'(x', y')\,\mathrm{d}x'^2 + 2B'(x', y')\,\mathrm{d}x'\mathrm{d}y' + C'(x', y')\,\mathrm{d}y'^2,$$

where A', B' and C' can be calculated. The geometry of the surface is the same however the co-ordinate lines are painted on — as an example take $\mathbb{R}^2$ with the metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$.

The cylinder in $\mathbb{R}^3$, $x^2 + y^2 = R^2$ has the intrinsic metric $\mathrm{d}s^2 = \mathrm{d}z^2 + R^2\,\mathrm{d}\phi^2$ (using cylindrical polars). We can locally make the co-ordinate change $x = z, y = R\phi$ and we get the flat space metric. The intrinsic geometry of the cylinder is that of a plane, although the cylinder has extrinsic curvature.

We can do the same sort of thing with the 2-sphere of radius a in $\mathbb{R}^3$. This has the intrinsic metric $\mathrm{d}s^2 = a^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$ (where θ and ϕ are the usual spherical polars). Letting $r = a\theta$ we get the metric

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}r^2 + \sin^2 \frac{r}{a}\, \mathrm{d}\phi^2 \\ &\approx \mathrm{d}r^2 + \left(r^2 - \frac{r^4}{3a^2} + \dots\right) \mathrm{d}\phi^2 \quad \text{near the North pole.} \end{aligned}$$

These extra terms are the effects of the intrinsic curvature of the sphere, which is $K = a^{-2}$.

The circumference of the circle at constant r is

$$C = 2\pi \left(r - \frac{r^3}{6a^2} + \dots\right) = \oint \mathrm{d}s,$$

and the area within the circle is

$$A = \pi r^2 - \frac{\pi r^4}{12a^2} + \dots = \int C \mathrm{d}r.$$

We note that

$$K = \lim_{r\to 0} \frac{3}{\pi} \frac{2\pi r - C}{r^3} = \lim_{r\to 0} \frac{12}{\pi} \frac{\pi r^2 - A}{r^4}. \tag{1.1}$$

1.1.1 Geodesics

These are the generalisation of straight lines in a flat space. If points are not too far apart we can find the geodesic γ by extremizing the length $\int_\gamma \mathrm{d}s$.

Geodesics are intrinsic to the surface (they depend on the metric). As an example, great circles on the sphere are geodesics.

We can find the intrinsic curvature of any surface at a point P by drawing all geodesics from P out to a distance r. We evaluate the circumference $C(r)$ and thus area $A(r)$ and use (1.1) to define the curvature K at that point. K can be negative (for instance at a saddle).

Let $\xi(r)$ (the geodesic deviation) be the distance between the ends of two nearby geodesics of length r from a point P. On S^2, $\xi(r) = a \sin \frac{r}{a} \delta\phi$ and we note that ξ satisfies the *geodesic deviation equation*:

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}r^2} = -K\xi.$$

A similar equation holds in any curved space. Thus if $K > 0$ two neighbouring geodesics recross (eventually), if $K = 0$ the geodesics are straight lines and if $K < 0$ the geodesics separate exponentially.

1.2 The principle of equivalence

1.2.1 Uniqueness of free fall

Consider the (Newtonian) dynamics of a single particle under gravity,

$$\mathcal{M}\ddot{\mathbf{x}} = -M\nabla\phi,$$

where $\mathcal{M}$, the inertial mass, equals M, the passive gravitational mass. Motion under gravity is independent of mass and composition.

If a gravitational field $\mathbf{g} = -\nabla\phi$ is constant in space and time then all particles have a constant acceleration $\mathbf{a} = \mathbf{g}$ superimposed on the gravity-free motion, $\mathbf{x} = \mathbf{x}' + \frac{1}{2}\mathbf{a}t^2$, where $\mathbf{x}'$ could be regarded as the position in an inertial frame with no gravitational field. Conversely, uniform acceleration $\mathbf{a}$ applied to the co-ordinates gives the illusion of a uniform gravitational field $\mathbf{a}$. Uniform gravitational fields are "fictitious" — they can be eliminated by a change of co-ordinates.

In *any* gravitational field, if an observer falls freely in a non-rotating laboratory, he [1] sees objects in the laboratory moving essentially on straight lines — the local gravitational field has been eliminated.

A freely falling non-rotating laboratory provides a *local inertial frame* allowing inertial co-ordinates $(\mathbf{x}, t)$ to be set up near the laboratory.

There are limitations on local inertial frames. Nearby particles at $\mathbf{x}$ and $\mathbf{x} + \boldsymbol{\xi}$ have relative tidal acceleration

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}t^2} = -\phi_{,ij}\xi^j.$$

In a "true" non-uniform gravitational field tidal forces cannot be eliminated by co-ordinate transforms and there are many different local inertial frames with relative accelerations.

1.2.2 Equivalence principle

> *All local inertial frames are equivalent for the performance of all experiments. All non-gravitational laws of physics take their special-relativistic forms in local inertial frames (by the usual arguments of special relativity).*

We can thus do things like fluid dynamics, quantum mechanics and electromagnetic theory in a gravitational field by using the special-relativistic laws and local inertial frames.

The speed of light is therefore c and distances and times are measured by the Minkowski metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\mathrm{d}t^2$.

1.2.3 Consequences for light propagation

The obvious consequence is that light can be deflected by gravitational fields (just like ordinary matter) because light moves in straight lines in local inertial frames which accelerate with respect to global co-ordinates.

There is also a gravitational frequency shift. Consider a lift of height h accelerating downwards at a rate g with respect to the earth. The lift has speed 0 at $t = 0$ and a light ray of frequency ν is emitted from the base of the lift at $t = 0$. At $t = hc^{-1}$ the light ray is at the top of the lift and has an observed frequency ν in the lift frame (by the equivalence principle). The lift then has speed $\frac{gh}{c}$ and so the light has a Doppler shifted frequency $\nu\left(1 - \frac{gh}{c^2}\right)$ measured from the earth frame. Note that

$$\frac{\mathrm{d}\nu}{\nu} = -\frac{\mathrm{d}\phi}{c^2}. \tag{1.2}$$

[1] More properly, (s)he. Sex will be assigned at random.

The same thing happens for light emitted in the other direction. [2]

We can integrate (1.2) to find

$$\frac{\nu}{\nu_0} = \exp\left(\frac{\phi_0 - \phi}{c^2}\right)$$

for a photon emitted at P_0 with a frequency ν_0 and observed with a frequency ν at P.

We expect that clocks in a potential well will appear to go slow (gravitational time dilation). This is observed for spectral lines in some white dwarf stars, but is not a big effect.

1.2.4 Special relativity and gravitation

Can we fix up special relativity so that it holds over an extended region containing gravitational fields?

Gravitational time dilation implies that a good clock at rest measures a time $t_m = t_c \exp\frac{\phi}{c^2}$, where (x_c, y_c, z_c, t_c) are special relativistic co-ordinates. We can only make the theory Lorentz invariant if all measurements obey

$$\mathrm{d}s_m^2 = \exp\left(\frac{\phi}{c^2}\right)\,\mathrm{d}s_c^2.$$

This is completely equivalent to a special type of curved spacetime theory (*not* GR) with metric

$$\mathrm{d}s^2 = \exp\left(\frac{\phi}{c^2}\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\mathrm{d}t^2\right),$$

and this is a more natural viewpoint. We see that attempts at combining special relativity and Newtonian gravity lead naturally to curved spacetimes.

1.3 Outline of general relativity

Our arguments have led us to a curved spacetime with four co-ordinates x^a ($a = 1 \dots 4$) and a metric $\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$, where $g_{ab} = g_{ba}$ depends on position.

At any event P in spacetime one can find a local inertial frame — one can make a co-ordinate change such that

$$g_{ab} = \eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c^2 \end{pmatrix}.$$

η_{ab} is the *Minkowski metric* (here defined oppositely to Electrodynamics).

The metric g_{ab} has the canonical form $+ + + -$ at each point.

Recall that intervals with $\mathrm{d}s^2 < 0$ are *timelike*, intervals with $\mathrm{d}s^2 > 0$ are *spacelike* and intervals with $\mathrm{d}s^2 = 0$ are *null*.

[2] This aspect of the equivalence principle is confirmed by the Pound-Rebka experiment.

Ordinary massive bodies move on timelike paths $x^a(\lambda)$ with $\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$ timelike and light travels on null paths with $\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$ null.

We also have the *clocks hypothesis*: the metric determines the time measured by a standard clock moving on a timelike path $x^a(\lambda)$ from event A to event B:

$$\tau = \frac{1}{c}\int_A^B |\mathrm{d}s|\,.$$

The *length hypothesis* is that standard rods measure the length $\mathrm{d}s$.

As for the equations of motion, we know that particles move on essentially straight lines in local inertial frames and this suggests the *geodesic hypothesis*, that freely falling particles move on geodesics. The path of a particle moving from A to B extremizes

$$\int_A^B |\mathrm{d}s|\,.$$

We guess that massive particles move on timelike geodesiscs and that massless particles move on null geodesics.

1.4 Static spacetime and Newtonian gravity

We compare general relativity with Newtonian gravity for low speeds and weak fields.

1.4.1 Static metrics

Consider a time independent gravitational field produced by a system of bodies at rest. We have co-ordinates $x^a = (x^i, t)$ $(i = 1, 2, 3)$ and the metric tensor g_{ab} is a function of the x^i alone. This is a stationary metric, and allows us to synchronize clocks.

Observer 1 at x_1^i bounces light rays off observer 2 at x_2^i. Since the metric is independent of time the photon paths $x^i(\lambda)$ in space, and hence the time t elapsed must be the same for each bounce.

Observer 1 sees the photon returning regularly after a proper time interval τ_1 and observer 2 sees the photon returning regularly after a proper time interval τ_2. Thus observer 2 can measure time by defining $t_2 = \frac{k_1}{k_2}\tau_2 +$ const. Observers at all points in space can do the same thing — ensure that time passes at the same rate.

However, can we actually synchronize the origin of time for different observers? We can do this (as shown) if the metric is *static*:

$$\mathrm{d}s^2 = g_{ij}(x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j - A(x^k)\,\mathrm{d}t^2.$$

In this case the metric is symmetric under time reversal and so the time reverse of a photon path is also a photon path.

If the field is produced by matter at rest then the matter distribution is invariant under time reversal and so the metric should also be symmetric — i.e. static.

Stationary metrics are produced by steadily moving distributions of matter — for example rotating stars.

From gravitational time dilation we see that $A = c^2 \exp\left(\frac{2\phi}{c^2}\right)$.

1.4.2 Newtonian limit

Take a static metric and a weak field, that is $\left|\frac{\phi}{c^2}\right|$ small. Then $A \approx c^2 + 2\phi$ and we expect $g_{ij} = \delta_{ij}\mathcal{O}(\frac{\phi}{c^2})$.

Consider a slowly moving particle with $v^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}$ such that $v^2 \ll c^2$. We thus have

$$\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = \left(v^2 + \mathcal{O}(v^2\tfrac{\phi}{c^2}) - c^2 - 2\phi + \mathcal{O}(\tfrac{\phi^2}{c^2})\right)\mathrm{d}t^2 \quad \text{on the path.}$$

Thus

$$\int |\mathrm{d}s| \approx \int c^2 + \phi - \tfrac{1}{2}v^2\,\mathrm{d}t$$

and the Euler-Lagrange equations yield $\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\frac{\partial\phi}{\partial x^i}$.

Chapter 2

Metric differential geometry

We need a formulation of physics valid in arbitrary co-ordinate systems. Physical quantities must have existence independent of particular co-ordinates being used – hence must transform properly under co-ordinate transforms. They should be represented by tensors.

2.1 Basic tensors

Consider the co-ordinate change $x^a \mapsto x^{a'}(x^b)$ on spacetime, with inverse $x^{a'} \to x^a(x^{b'})$.

Define

$$p_a^{a'} = \frac{\partial x^{a'}}{\partial x^a}, \quad p_{a'}^{a} = \frac{\partial x^a}{\partial x^{a'}}.$$

Note that $p_a^{a'} p_{a'}^b = \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^b}{\partial x^{a'}} = \delta_a{}^b$, by using the chain rule, where

$$\delta_a{}^b = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

is the Kronecker delta.

Under repeated co-ordinate change $x^a \mapsto x^{a'} \mapsto x^{a''}$, we have the group property, using the chain rule,

$$p_a^{a'} p_{a'}^{a''} = p_a^{a''}.$$

A *covariant tensor* of n^{th} rank, with components $T_{a\cdots b}$ with respect to co-ordinates x^a, at a point P has transformation law

$$T_{a\cdots b} \to T_{a'\cdots b'} = p_{a'}^a \cdots p_{b'}^b T_{a\cdots b}.$$

Note that the group property shows that the components $T_{a'\cdots b'}$ are uniquely defined with respect to any co-ordinate system if they are fixed in one system x^a; this provides a way of constructing all tensors.

A *contravariant tensor* $T^{a\ldots b}$ transforms as

$$T^{a\cdots b} \to T^{a'\cdots b'} = p_a^{a'} \ldots p_b^{b'} T^{a\cdots b}$$

Similarly for *mixed tensors*, for example

$$T_a{}^b \to T_{a'}{}^{b'} = p_{a'}^a p_b^{b'} T_a{}^b$$

It is important to keep the order of the indices the same.

A *scalar* is a tensor with no indices, invariant under co-ordinate change, for example the mass of a particle.

A *scalar field* $\phi(x^a)$ is a scalar function for example pressure or particle density in a fluid.

A *covariant vector field* $v_a(x^b)$ is a vector function of position. For example if $\phi(x^b)$ is a scalar field then $v_a = \phi_{,a} := \frac{\partial\phi}{\partial x^a}$ is a covariant vector field, since

$$v_{a'} = \frac{\partial\phi}{\partial x^{a'}} = \frac{\partial x^a}{\partial x^{a'}}\frac{\partial\phi}{\partial x^a}.$$

A further example is a pressure gradient $p_{,a}$ in a fluid.

Suppose a curve $x^a(\lambda)$ parametrised by λ, has a tangent of $v^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$ at the point P. v^a is a *contravariant vector*. This follows because

$$v^{a'} = \frac{\mathrm{d}x^{a'}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^{a'}}{\mathrm{d}x^a}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}.$$

Other examples are the 4-velocity of an observer, $u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}$, where τ is proper time.

2.1.1 Examples of tensors

- The Kronecker delta is a tensor:

$$p_{a'}^a p_b^{b'} \delta_a{}^b = p_{a'}^a p_a^{b'} = \delta_{a'}{}^{b'}.$$

- The metric g_{ab} is a tensor, since the invariant $\mathrm{d}s^2$ can be written

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = g_{ab}\frac{\partial x^a}{\partial x^{a'}}\mathrm{d}x^{a'}\frac{\partial x^b}{\partial x^{b'}}\mathrm{d}x^{b'} = g_{a'b'}\mathrm{d}x^{a'}\mathrm{d}x^{b'}$$

  where $g_{a'b'} = g_{ab}p_{a'}^a p_{b'}^b$.

Further examples will be provided by the curvature and energy momentum tensors.

2.1.2 Operations preserving tensor property

- Addition: $T_{ab} + W_{ab}$ is a tensor if T_{ab} and W_{ab} are tensors.
- Scalar multiplication: fT_{ab} is a tensor if T_{ab} is a tensor.
- Outer products: v^aT_{bc} transforms as

$$v^{a'}T_{b'c'} = p_a^{a'} p_{b'}^b p_{c'}^c v^a T_{bc}.$$

- Contraction of one covariant with one contravariant index: if $T^{a}{}_{bc}$ is a tensor, define $v_c = T^a{}_{ac}$, transforming as

$$\begin{aligned} v_{c'} &= T^{a'}{}_{a'c'} \\ &= p^{a'}_{a} p^{b}_{a'} p^{c}_{c'} T^{a}{}_{bc} = \delta_a{}^{b} p^{c}_{c'} T^{a}{}_{bc} \\ &= p^{c}_{c'} T^{a}{}_{ac} = p^{c}_{c'} v_c . \end{aligned}$$

- Interchange of indices: T_{ab} (a tensor) $\mapsto T_{ba}$ which is also a tensor. Similarly symmetrisation and anti-symmetrisation

$$\begin{aligned} T_{(ab)} &= \frac{1}{2!}(T_{ab} + T_{ba}) \\ T_{[ab]} &= \frac{1}{2!}(T_{ab} - T_{ba}) \end{aligned}$$

The above can readily be generalised to more indices.

2.1.3 Quotient theorem

Suppose $U^a = T^{ab}V_b$ is a vector for all vectors V_b. Then $p^{a'}_{a} U^a = p^{a'}_{a} T^{ab} V_b$, and

$$U^{a'} = p^{a'}_{a} U^a = T^{a'b'} V_{b'} = T^{a'b'} p^{b}_{b'} V_b .$$

Subtracting these last two yields,

$$\begin{aligned} (T^{a'b'} p^{b}_{b'} - p^{a'}_{a} T^{ab}) V_b = 0 \qquad \forall V_b \text{ and so} \\ T^{a'b'} p^{b}_{b'} = p^{a'}_{a} T^{ab} . \end{aligned}$$

Multiplying both sides by $p^{c'}_{b}$ yields

$$T^{a'b'} p^{b}_{b'} p^{c'}_{b} = T^{a'b'} \delta_{b'}{}^{c'} = T^{a'c'} = p^{c'}_{b} p^{a'}_{a} T^{ab} .$$

Hence T^a is a tensor.

2.1.4 Inverse metric tensor

Define $g^{ab} (= g^{ba})$ to be the matrix inverse of g_{ab}, i.e. such that $g_{ac} g^{cb} = \delta_a{}^b$. Now for any vector V^a can define a vector $U_a = g_{ab} V^b$. Note that there is a one to one correspondance between U_a and V^a since g_{ab} is non-singular and so we can construct all vectors U_b in this way. The quotient theorem implies that g^{ab} is a tensor.

2.1.5 Raising and lowering of indices

We can use g^{ab} to raise any covariant index. For example T_{ab} gives a tensor $T^a{}_b = g^{ac} T_{cb}$ if the first index is raised. Similarly we can use g_{ab} to lower any index, for example W^{ab} gives $W_a{}^b = g_{ac} W^{cb}$. The index ordering must be carefully maintained.

Raising and lowering are inverse operations. One normally regards e.g. T_{ab}, $T_a{}^b$, $T^a{}_b$ and T^{ab} as different versions of the same object.

2.1.6 Partial derivatives of tensors

Partial derivatives of tensors are not tensors in general. For example suppose v_a is a vector field. Then $v_{a'} = \frac{\partial x^a}{\partial x^{a'}} v_a$ and so

$$\begin{aligned}\frac{\partial v_{a'}}{\partial x^{b'}} &= \frac{\partial^2 x^a}{\partial x^{a'} \partial x^{b'}} v_a + \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} \frac{\partial v_a}{\partial x^b} \\ &\neq p^a_{a'} p^b_{b'} \frac{\partial v_a}{\partial x^b} \qquad \text{in general.}\end{aligned}$$

The only exception is $\phi_{,a}$ as mentioned earlier.

2.2 Lengths and geodesics

The squared magnitude of a vector v_a or v^a is defined to be $v_a v^a = v_a v_b g^{ab} = v^a v^b g_{ab}$ and is invariant under co-ordinate transformations.

It can be evaluated in a local inertial frame where $g_{ab} = \eta_{ab}$.

$$v_a \text{ is } \begin{cases} \text{spacelike if } v_a v^a > 0, \\ \text{null if } v_a v^a = 0, \\ \text{timelike if } v_a v^a < 0. \end{cases}$$

As before, if v^a is spacelike we can find a Lorentz transformation in the local inertial frame making $v^a = (v^i, 0)$. Then $v_a v^a = (v^1)^2 + (v^2)^2 + (v^3)^2 = |\mathbf{v}|^2$, which is the physically measured squared magnitude of v^a in that frame. If v^a is timelike we can make $v^a = (0, v^4)$ and then $v_a v^a = -c^2 (v^4)^2$.

2.3 Angles between vectors

Suppose that v^a and w^a are both spacelike and that we have chosen a local inertial frame such that $v^a = (v^i, 0)$, $w^a = (w^i, 0)$. Then the angle θ between v^a and w^b is defined by

$$\begin{aligned}\cos\theta &= \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|} \qquad \text{Cartesian notation in LIF, or} \\ &= \frac{(g_{ab} v^a w^b)}{(v_d v^d)^{1/2} (w_c w^c)^{1/2}} \qquad \text{invariant definition.}\end{aligned}$$

2.4 Lengths of curves

If $x^a(\lambda)$ describes a spacelike curve γ, which is parameterised by λ (i.e. if $v^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$ is spacelike along γ), the length of γ from A to B is

$$\int_A^B \mathrm{d}s = \int_A^B \left(g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} \right)^{1/2} \mathrm{d}\lambda.$$

If $x^a(\lambda)$ gives a timelike curve γ (i.e. $v^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$ is timelike along γ), then the time elapsed along γ from A to B is

$$\frac{1}{c}\int_A^B |\mathrm{d}s| = \frac{1}{c}\int_A^B \left(-g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\right)^{1/2} \mathrm{d}\lambda.$$

2.5 Geodesics

A geodesic γ from A to B extremises

$$\int_A^B |\mathrm{d}s| = \int_A^B \left|g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\right|^{1/2} \mathrm{d}\lambda = \int_A^B L(x^a(\lambda), \dot{x}^b(\lambda))\,\mathrm{d}\lambda,$$

where $\dot{x}^a(\lambda) = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$, subject to fixed endpoints: $x^a(\lambda_1)$ are the co-ordinates of A, and $x^a(\lambda_2)$ are the co-ordinates of B.

For example consider a spacelike geodesic. Then

$$\frac{\partial L}{\partial \dot{x}^a} = \frac{g_{ab}\dot{x}^b}{L}$$
$$\frac{\partial L}{\partial x^a} = \frac{g_{bc,a}\dot{x}^b\dot{x}^c}{2L}$$

Using the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0$$

gives

$$L^{-1}\left[g_{ab}\ddot{x}^b + (g_{ab,c} - \frac{1}{2}g_{bc,a})\dot{x}^b\dot{x}^c\right] = L^{-2}\frac{\mathrm{d}L}{\mathrm{d}\lambda}g_{ab}\dot{x}^b.$$

Using our freedom to reparametrise the curve we can choose $\lambda = s$, the distance along γ. Then $L = 1$ and $\frac{\mathrm{d}L}{\mathrm{d}\lambda} = 0$ along γ. Therefore

$$\begin{aligned} 0 &= g_{ab}\ddot{x}^b + (g_{ab,c} - \frac{1}{2}g_{bc,a})\dot{x}^b\dot{x}^c \\ &= g_{ab}\ddot{x}^b + \frac{1}{2}(g_{ab,c} + g_{ac,b} - g_{cb,a})\dot{x}^b\dot{x}^c. \end{aligned}$$

Raising index a yields the *geodesic equation*

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}s^2} + \left\{ {a \atop bc} \right\}\frac{\mathrm{d}x^b}{\mathrm{d}s}\frac{\mathrm{d}x^c}{\mathrm{d}s} = 0,$$

where

$$\left\{ {a \atop bc} \right\} = \frac{1}{2}g^{ad}(g_{bd,c} + g_{cd,b} - g_{bc,d}).$$

The same equation is obtained for timelike geodesics and we have the equations of motion for a test particle in a gravitational field.

The expression $\left\{ {a \atop bc} \right\}$ involves the "derivatives of gravitational potential" and corresponds to $\phi_{,i}$ in Newtonian gravity. It is possible to rederive Newtonian dynamics in this way – see later.

The geodesic equation is a second order ordinary differential equation and so a geodesic is uniquely specified once the starting point $x^a(0)$ and an initial tangent direction $\dot{x}^a(0)$ are chosen.

2.6 Covariant differentiation and Christoffel symbols

Physical laws involve partial derivatives. We need a generalisation ∇_a of $\partial_a := \frac{\partial}{\partial x^a}$ which preserves tensorial properties. We want the covariant derivative operator to

- keep $\nabla_a \phi = \partial_a \phi$ for scalar fields ϕ, since $\partial_a \phi$ is already a covariant vector field.
- look like $\nabla_b v_a = \partial_b v_a - \Gamma^c_{ba} v_c$ acting on covariant vector fields, where Γ^c_{ba} is a (non-tensorial) collection of 4^3 numbers to be constructed out of the metric and its first derivatives, and $-\Gamma^c_{ba} v_c$ is designed to cancel out the bad transformation properties of $\partial_b v_a$.
- commute with addition:

$$\nabla_d (T^{a\ldots b}{}_{d\ldots c} + U^{a\ldots b}{}_{d\ldots c}) = \nabla_d T^{\cdots}{}_{\ldots} + \nabla_d U^{\cdots}{}_{\ldots}.$$

- obey the Leibniz rule

$$\nabla_a \left(T^{\cdots}{}_{\ldots} U^{\cdots}{}_{\ldots} \right) = \left(\nabla_a T^{\cdots}{}_{\ldots} \right) U^{\cdots}{}_{\ldots} + T^{\cdots}{}_{\ldots} \left(\nabla_a U^{\cdots}{}_{\ldots} \right)$$

- satisfy $\nabla_d g_{ab} = 0$, $\nabla_d g^{ab} = 0$ and $\nabla_d \delta^a{}_b = 0$.
- commute with index contraction:

$$\nabla_a \left(T^{\cdots b \cdots}{}_{\cdots b \cdots} \right) = \delta^d{}_c \nabla_a \left(T^{\cdots c \cdots}{}_{\cdots d \cdots} \right).$$

These properties imply that ∇_a commutes with the operations of raising/lowering indices.

We want to find the Γ's, which we will do using the zero covariant derivative of the metric. First note that for any covariant vector fields u_a and v_b,

$$\begin{aligned} \nabla_d (u_a v_b) &= u_a \nabla_d v_b + v_b \nabla_d u_a \\ &= \partial_d (u_a v_b) - \Gamma^c_{db} u_a v_c - \Gamma^c_{da} u_c v_b. \end{aligned}$$

Now any tensor field T_{ab} can be built by adding tensors of the form $u_a v_b$, so using linearity

$$\nabla_d T_{ab} = \partial_d T_{ab} - \Gamma^c_{db} T_{ac} - \Gamma^c_{da} T_{cb}.$$

for any tensor T_{ab}. We apply this to the metric tensor g_{ab} to get

$$\nabla_d g_{ab} = g_{ab,d} - \Gamma^c_{db} g_{ac} - \Gamma^c_{da} g_{cb} = 0. \tag{2.1}$$

Permuting the indices cyclically, we get

$$\nabla_a g_{bd} = g_{bd,a} - \Gamma^c_{ad} g_{bc} - \Gamma^c_{ab} g_{cd} = 0. \tag{2.2}$$
$$\nabla_b g_{da} = g_{da,b} - \Gamma^c_{ba} g_{dc} - \Gamma^c_{bd} g_{ca} = 0. \tag{2.3}$$

We make the further simplifying assumption of symmetry: $\Gamma^{\ c}_{ab} = \Gamma^c_{ba}$. Now take (2.3) – (2.1) – (2.2) and adjust the indices to get

$$2\Gamma^d_{bc} g_{ad} = -g_{bc,a} + g_{ac,b} + g_{ab,c}.$$

Raise the index a to get

$$\Gamma^a_{bc} = \tfrac{1}{2} g^{ad} (g_{bd,c} + g_{cd,b} - g_{bc,d}). \tag{2.4}$$

These are the *Christoffel symbols* for the metric g_{ab} and define the metric connection ∇ on spacetime.

2.6.1 Transformation properties of Γ

We start from $g_{a'b'} = p^a_{a'} p^b_{b'} g_{ab}$, so that

$$g_{a'b',c'} = p^a_{a'} p^b_{b'} p^c_{c'} g_{ab,c} + g_{ab} \partial_{c'} \left(\frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} \right).$$

$$\begin{aligned} g_{a'b',c'} + g_{a'c',b'} - g_{b'c',a'} &= p^a_{a'} p^b_{b'} p^c_{c'} \left(g_{ab,c} + g_{ac,b} - g_{bc,a} \right) \\ &+ g_{ab} \left(\partial_{c'} \left(\frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} \right) + \partial_{b'} \left(\frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^c}{\partial x^{c'}} \right) - \partial_{a'} \left(\frac{\partial x^b}{\partial x^{b'}} \frac{\partial x^c}{\partial x^{c'}} \right) \right) \end{aligned}$$

Putting all of this together we find

$$\Gamma^{a'}_{b'c'} = p^{a'}_a p^b_{b'} p^c_{c'} \Gamma^a_{bc} + \frac{\partial x^{a'}}{\partial x^a} \frac{\partial^2 x^a}{\partial x^{b'} \partial x^{c'}}. \tag{2.5}$$

(2.5) can be used to verify that $\nabla_b v_a$ is a tensor.

2.6.2 Action of ∇_b on other types of tensor

For instance: what is $\nabla_b u^a$? Take an arbitrary covariant vector field v_a and consider $\nabla_b (v_a u^a) = \partial_b (v_a u^a)$. Then

$$\begin{aligned} \nabla_b (v_a u^a) &= u^a \nabla_b v_a + v_a \nabla_b u^a \\ &= u^a \partial_b v_a - \Gamma^c_{ba} v_c u^a + v_a \nabla_b u^a \\ &= u^a \partial_b v_a + v_a \partial_b u^a. \end{aligned}$$

This is true for all v_a, so that

$$\nabla_b u^a = \partial_b u^a + \Gamma^a_{bc} u^c.$$

In general we get a $+$ sign for each contravariant index and a $-$ sign for each covariant index, that is

$$\nabla_b T_a^{\ c} = \partial_b T_a^{\ c} - \Gamma^d_{ba} T_d^{\ c} + \Gamma^c_{bd} T_a^{\ d}.$$

We write ∇_b ( ) as $(\ )_{;b}$.

2.7 Differentiation along a curve: geodesics

We need a geometrical description of the rate of change of a physical quantity seen by an observer moving along a path $x^a(\lambda)$. This is the *absolute derivative*, given by

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{d}\lambda}v^a &= \frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\nabla_b v^a = \frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\frac{\partial v^a}{\partial x^b} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}v^c \\ &= \frac{\mathrm{d}v^a}{\mathrm{d}\lambda} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}v^c.\end{aligned}$$

Note that we only need to know v^a along the path. We can similarly define $\frac{\mathrm{D}}{\mathrm{d}\lambda}$ on other fields. The absolute derivative of a tensor is again a tensor.

A field v^a is said to be *parallelly transported* along a curve $x^a(\lambda)$ iff $\frac{\mathrm{D}v^a}{\mathrm{d}\lambda} = 0$ (and similarly for other types of tensor).

Note that parallel transport preserves lengths and angles. If v^a and w^a are parallelly transported, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(v_a w^a\right) &= \frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\nabla_b\left(g_{cd}v^c w^d\right) \\ &= \frac{\mathrm{d}x^b}{\mathrm{d}\lambda}g_{cd;b} + g_{cd}w^d\frac{\mathrm{D}v^c}{\mathrm{d}\lambda} + g_{cd}v^c\frac{\mathrm{D}w^c}{\mathrm{d}\lambda} \\ &= 0.\end{aligned}$$

We can apply the notation of absolute derivative to the tangent vector $\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$. A curve $x^a(\lambda)$ is said to be *autoparallel* iff

$$\frac{\mathrm{D}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda} = 0,$$

that the tangent vector is parallelly transported along the curve. This is equivalent to

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\lambda^2} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\frac{\mathrm{d}x^c}{\mathrm{d}\lambda} = 0,$$

which, since $\Gamma^a_{bc} = \left\{{a \atop bc}\right\}$, is the geodesic equation. This gives an alternative characterisation of geodesics, and λ is called an *affine parameter* along the geodesic.

If γ is a geodesic with affine parameter λ then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\right) = 0,$$

so that $g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}$ is a constant along γ and λ is proportional to length s (or proper time τ) along γ.

The acceleration (vector) of a timelike curve $x^a(\tau)$ with 4-velocity $u^b = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}$ is

$$a^b = \frac{\mathrm{D}u^b}{\mathrm{d}\tau} = \frac{\mathrm{d}^2x^b}{\mathrm{d}\tau^2} + \Gamma^b_{cd}\frac{\mathrm{d}x^c}{\mathrm{d}\tau}\frac{\mathrm{d}x^d}{\mathrm{d}\tau}.$$

and so geodesics are unaccelerated curves (free fall).

2.8 Local inertial frames

We can now make our definition of a LIF more precise. We want to choose locally inertial co-ordinates x^a near an event P ($x^a = 0$) such that $g_{ab} = \eta_{ab}$ at P and that particles moving through P under gravity have no co-ordinate acceleration. We want to arrange $\Gamma^a_{bc} = 0$ (or equivalently $g_{ab,c} = 0$) at P

In a LIF

- the metric looks as much as possible like the flat space metric
- geodesics become straight lines
- parallel transport, acceleration etc. acquire usual flat-space interpretations
- covariant derivatives become partial derivatives.

To find inertial co-ordinates near P we translate to put $x^a = 0$ and then use a linear transformation to give $g_{ab} = \eta_{ab}$ at P. Define ${}^0\Gamma^a_{bc} = \Gamma^a_{bc}|_P$. Then use the transformation $x^a \to y^a$ with quadratic inverse

$$x^a = y^a - \tfrac{1}{2}{}^0\Gamma^a_{bc} y^b y^c.$$

In the new co-ordinates,

$$\begin{aligned} g^{\text{new}}_{ab} &= \frac{\partial x^c}{\partial y^a}\frac{\partial x^d}{\partial y^b} g^{\text{old}}_{cd} \\ &= \left(\delta_a{}^c - {}^0\Gamma^c_{ae} y^e\right)\left(\delta_b{}^d - {}^0\Gamma^d_{bf} y^f\right)\left(\eta_{cd} + g^{\text{old}}_{cd,g} y^g + \dots\right). \end{aligned}$$

The terms linear in y^c are

$$\left(-{}^0\Gamma^d_{ac}\eta_{bd} - {}^0\Gamma^d_{bc}\eta_{ad} + g^{\text{old}}_{ab,c}\right) y^c = 0.$$

Hence $g^{\text{new}}_{ab} = \eta_{ab} +$ quadratic in y^c — the co-ordinates y^a provide a LIF near P.

2.9 Curvature

The curvature of spacetime measures the non-commutation of covariant derivatives. For a scalar field ϕ, $\phi_{;ab} = \phi_{;ba}$, but for a vector field v^a,

$$\begin{aligned} v^a_{;bc} - v^a_{;cb} &= \left(\Gamma^a_{be,c} - \Gamma^a_{ce,b} + \Gamma^a_{cd}\Gamma^d_{be} - \Gamma^a_{bd}\Gamma^d_{ce}\right) v^e \\ &= R^a{}_{ecb} v^e, \end{aligned}$$

where

$$R^a{}_{ecb} = \Gamma^a_{be,c} - \Gamma^a_{ce,b} + \Gamma^a_{cd}\Gamma^d_{be} - \Gamma^a_{bd}\Gamma^d_{ce}.$$

$R^a{}_{ecb}$ is a tensor (by the quotient theorem) and is called the the *Riemann curvature tensor*. It is constructed from the metric and its first and second covariant derivatives. If the spacetime is flat we can choose Minkowskian co-ordinates to get $g_{ab} = \eta_{ab}$ so that $R^a{}_{bcd} = 0$. Therefore $R^a{}_{bcd} = 0$ in all co-ordinates. The converse can be proved: if $R^a{}_{bcd} = 0$ then the spacetime is flat.

In a LIF at P,

$$R_{abcd} = \tfrac{1}{2}\left(g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}\right).$$

This gives the symmetry properties

- $R_{abcd} = R_{[ab]cd} := \frac{1}{2}\left(R_{abcd} - R_{bacd}\right)$
- $R_{abcd} = R_{ab[cd]}$
- $R_{abcd} = R_{cdab}$
- $R_{a[bcd]} := \frac{1}{3!}\left(R_{abcd} + R_{acdb} + R_{adbc} - R_{acbd} - R_{abdc} - R_{adcb}\right) = 0$. Using the other symmetries of R_{abcd}, this can be equivalently written as

$$R_{abcd} + R_{acdb} + R_{adbc} = 0.$$

Since symmetries of tensors are preserved by co-ordinate transformations, these hold at any point P in any co-ordinates. These symmetries imply that R_{abcd} has only 20 free components.

The *Ricci tensor* is $R_{bd} = R^{a}{}_{bad}$. Note that

$$R_{bd} = g^{ac}R_{abcd} = g^{ac}R_{cdab} = R_{db}.$$

R_{bd} therefore has only ten free components. The *Ricci scalar* is $R = g^{bd}R_{bd}$.

2.10 Geodesic deviation

Spacetime curvature produces relative acceleration of nearby test particles moving on geodesics. For convenience in the derivation we replace “2 nearby test particles” with “1 parameter family of geodesics”. Each geodesic is labelled by a parameter s. We label points on a given geodesic by proper time τ measured from the origin.

Write $u^a = \frac{\partial}{\partial\tau}x^a(\tau, s)$: the 4-velocity on the geodesic labelled by s. The geodesic equation is

$$\frac{\mathrm{D}}{\partial\tau}u^a := \frac{\partial x^b}{\partial\tau}\nabla_b u^a = u^b\nabla_b u^a = 0.$$

Define $\xi^a = \frac{\partial}{\partial s}x^a(\tau, s)$. Then for small Δs, $\Delta s\xi^a$ is a separation vector from the geodesic labelled by s to the geodesic labelled by $s + \Delta s$.

Note that

$$\frac{\partial u^a}{\partial s} = \frac{\partial^2 x^a}{\partial s\partial\tau} = \frac{\partial\xi^a}{\partial\tau},$$

and so

$$\xi^b \nabla_b u^a = \frac{\partial x^b}{\partial s} \nabla_b u^a = \frac{\partial u^a}{\partial s} + \Gamma^a_{bc} \xi^b u^c$$
$$= \frac{\partial \xi^a}{\partial \tau} + \Gamma^a_{bc} u^b \xi^c = \frac{\partial x^b}{\partial \tau} \nabla_b \xi^a$$
$$= u^b \nabla_b \xi^a.$$

We now prove (and then use!) the *curvature identity*, which is valid for any vector fields X^a, Y^b and Z^c:

$$Y^b \nabla_b \left(Z^c \nabla_c X^a \right) - Z^c \nabla_c \left(Y^b \nabla_b X^a \right) =$$
$$Y^b \left(\nabla_b Z^c \right) \left(\nabla_c X^a \right) + Y^b Z^c \nabla_b \nabla_c X^a$$
$$- Z^c \left(\nabla_c Y^b \right) \left(\nabla_b X^a \right) - Z^c Y^b \nabla_c \nabla_b X^a$$
$$= \left(Y^c \nabla_c Z^b - Z^c \nabla_c Y^b \right) \nabla_b X^a + Y^b Z^c R^a{}_{dbc} X^d.$$

Now take $X^a = u^a$, $Y^b = u^b$ and $Z^c = \xi^c$, so that

$$Z^c \nabla_c X^a = \xi^c \nabla_c u^a = u^c \nabla_c \xi^a$$
$$Y^b \nabla_b X^a = u^b \nabla_b u^a = 0$$
$$Y^c \nabla_c Z^b - Z^c \nabla_c Y^b = u^c \nabla_c \xi^b - \xi^c \nabla_c u^b = 0.$$

Substituting these into the curvature identity we get

$$u^b \nabla_b \left(u^c \nabla_c \xi^a \right) = R^a{}_{dbc} u^d u^b \xi^c,$$

or

$$\frac{\mathrm{D}^2}{\partial \tau^2} \xi^a = R^a{}_{dbc} u^d u^b \xi^c. \tag{2.6}$$

This is the *equation of geodesic deviation*. It shows that the relative acceleration is proportional to separation for two nearby test bodies. We have a *true* gravitational field iff we have relative accelerations, iff $R^a{}_{bcd} \neq 0$, iff spacetime is curved.

Chapter 3

Vacuum gravitational fields

3.1 The vacuum field equations

We need to guess the field equations of General Relativity. We will use the Newtonian limit to suggest the vacuum GR field equations and then compare the predictions of these equations in the non-Newtonian case.

In the Newtonian limit (weak fields and low speeds), $\left|\frac{\phi}{c^2}\right| \ll 1$ and $\frac{V}{c} \ll 1$ (where V is a "typical" speed). We will use co-ordinates $x^a = (x, y, z, ct)$. From the equivalence principle, $g_{44} = -1 - \frac{2\phi}{c^2} + \dots$ and it is reasonable to expect that *all* deviations from flatness are of order $\frac{\phi}{c^2}$.

A geodesic in Newtonian gravity has $\tau \approx t$, $\left|\frac{\mathrm{d}x^i}{\mathrm{d}t}\right| \ll c$, and the spatial component of the geodesic equation is

$$0 = \frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} + \Gamma^i_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau} \approx \frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} + \Gamma^i_{44}c^2.$$

Now

$$\Gamma^i_{44} = \tfrac{1}{2}g^{i4}g_{44,4} + \tfrac{1}{2}g^{ij}(g_{j4,4} - g_{44,j}) \approx -\tfrac{1}{2}\delta^{ij}g_{44,j}$$

as $g^{ij} \approx \delta^{ij}$ and the derivative ∂_4 should be smaller than derivatives ∂_i by a factor of order $\mathcal{O}(\frac{v}{c}) \ll 1$. Thus $\Gamma^i_{44} \approx c^{-2}\phi_{,i}$ and the geodesic equation is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} \approx -\phi_{,i},$$

which is probably a good thing. Γ^i_{44} are the only Christoffel symbols significant for Newtonian gravity.

We now consider geodesic deviation in the Newtonian limit. Take a (spatial) separation vector $\xi^a = (\xi^i, 0)$. Now $u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau} = (0, c)$ at low speeds and the geodesic deviation equation

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{d}t^2} = R^a{}_{bdc}u^b u^d \xi^c$$

gives $\frac{\mathrm{d}^2\xi^i}{\mathrm{d}t^2} \approx c^2 R^i{}_{44j}\xi^j$. Therefore $R^i{}_{44j}$ are the only components of the Riemann tensor which are significant for Newtonian gravity. In the Newtonian limit

$$
\begin{aligned}
R^a{}_{bdc} &= \Gamma^a_{bc,d} - \Gamma^a_{bd,c} + \Gamma^a_{d\alpha}\Gamma^\alpha_{bc} - \Gamma^a_{c\alpha}\Gamma^\alpha_{bd} \\
&\approx \Gamma^a_{bc,d} - \Gamma^a_{bd,c}.
\end{aligned}
$$

Therefore $R^i{}_{44j} \approx \Gamma^i_{4j,4} - \Gamma^i_{44,j}$. If L is a typical lengthscale of the system then all the Γ^a_{bd} are $\mathcal{O}(\frac{\phi}{c^2 L})$ and so $\Gamma^i_{4j,4} = \mathcal{O}(\frac{\phi}{c^2 L^2}\frac{V}{c})$ but $\Gamma^i_{44,j} = \mathcal{O}(\frac{\phi}{c^2 L^2})$. Thus $R^i{}_{44j} \approx -\Gamma^i_{44,j} = -c^{-2}\phi_{,ij}$. The Newtonian geodesic deviation equation is therefore

$$
\frac{\mathrm{d}^2\xi^i}{\mathrm{d}t^2} \approx -\phi_{,ij}\xi^j.
$$

We base the vacuum GR equations on that for a Newtonian field, $\phi_{,ii} = 0$. In the Newtonian limit we find that

$$
R_{44} = R^i{}_{4i4} \approx c^{-2}\phi_{,ii} = 0 \text{ in vacuum.}
$$

Since we want *tensor* field equations valid in all co-ordinate systems this suggests

$$
R_{ab} = 0 \tag{3.1}
$$

for the vacuum field equations (vacuum Einstein equations). Since R_{ab} is symmetric we have 10 field equations, second order in the "gravitational potential" g_{ab}. A highly nonlinear gravitational field can act as its own source.

3.2 The Schwarzschild metric

We look for a solution of the vacuum Einstein equations describing the gravitational field outside a spherically symmetric body at rest. (It can be shown that) A static spherically symmetric metric has the form

$$
\mathrm{d}s^2 = e^{\alpha(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) - e^{\gamma(r)}\mathrm{d}t^2
$$

in suitable co-ordinates. There are no cross terms $\mathrm{d}t\mathrm{d}(\text{space})$ since the metric must be invariant under $t \to -t$.

The radial co-ordinate r is chosen for simplicity — such that each sphere with t, r constant has the intrinsic metric $r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$. We can change r to $r'(r)$, but the metric loses its simplicity in this case.

The spherical symmetry forbids cross terms $\mathrm{d}r\mathrm{d}\theta$ (etc) and makes g_{rr} a function of r only.

To impose the vacuum Einstein equations $R_{ab} = 0$ we need to find the Christoffel symbols. It is most convenient to find them via the geodesic equations. We use an alternate Lagrangian for the geodesics,

$$
\delta\int_A^B g_{ab}\dot{x}^a\dot{x}^b\,\mathrm{d}\lambda = 0,
$$

where $\dot{x}^a \equiv \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$. It is easy to use the Euler-Lagrange equations to show that this gives the geodesic equations. We find that λ must be a multiple of s or t along extremal curves.

In this spherically symmetric metric

$$\mathcal{L} = e^{\alpha(r)}\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - e^{\gamma(r)}\dot{t}^2. \tag{3.2}$$

The Euler-Lagrange equations give

$$2e^{\alpha}\ddot{r} + e^{\alpha}\alpha'\dot{r}^2 - 2r\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + e^{\gamma}\gamma'\dot{t}^2 = 0, \tag{3.3}$$

$$2r^2\ddot{\theta} + 4r\dot{r}\dot{\theta} - 2r^2\sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{3.4}$$

$$2r^2\sin^2\theta\ddot{\phi} + 4r\sin^2\theta\dot{\phi}\dot{r} + 4r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} = 0, \tag{3.5}$$

$$-2e^{\gamma}\ddot{t} - 2\gamma' e^{\gamma}\dot{r}\dot{t} = 0. \tag{3.6}$$

The only non-zero Christoffel symbols (where $x^a = (r, \theta, \phi, t)$) are:

$$\begin{array}{lll} \Gamma^1_{11} = \frac{1}{2}\alpha' & \Gamma^1_{22} = -re^{-\alpha} & \Gamma^1_{33} = -re^{-\alpha}\sin^2\theta \\ \Gamma^1_{44} = \frac{1}{2}\gamma' e^{\gamma-\alpha} & \Gamma^2_{12} = r^{-1} & \Gamma^2_{33} = -\sin\theta\cos\theta \\ \Gamma^3_{13} = r^{-1} & \Gamma^3_{23} = \cot\theta & \Gamma^4_{14} = \frac{1}{2}\gamma' \end{array}$$

and the transposes $\Gamma^a_{bc} = \Gamma^a_{cb}$. We can now find the Ricci tensor, which has non-zero components

$$R_{11} = -\tfrac{1}{2}\gamma'' + \tfrac{1}{4}\alpha'\gamma' + \tfrac{1}{4}\gamma'^2 + r^{-1}\alpha' \tag{3.7}$$

$$R_{22} = e^{-\alpha}\left(\tfrac{1}{2}r(\alpha' - \gamma') - 1\right) + 1 \tag{3.8}$$

$$R_{33} = \sin^2\theta R_{22} \tag{3.9}$$

$$R_{44} = e^{\gamma-\alpha}\left(\tfrac{1}{2}\gamma'' - \tfrac{1}{2}\alpha'\gamma' + \tfrac{1}{4}\gamma'^2 + r^{-1}\gamma'\right). \tag{3.10}$$

Equations (3.7) and (3.10) give us $\alpha + \gamma = \kappa$ (a constant). Substituting into (3.8) we get $e^{-\alpha} = 1 - \frac{a}{r}$ (and we can check that this is consistent with (3.7)). Thus

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - \frac{a}{r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) - e^{\kappa}\left(1 - \tfrac{a}{r}\right)\mathrm{d}t^2.$$

We normalize t to ordinary time as $r \to \infty$, so that $e^{\kappa} = c^2$. In the far field, $g_{tt} = -c^2 + \frac{2GM}{r}$ if the body has mass M. Thus $a = \frac{2GM}{c^2}$ and we arrive at the Schwarzschild metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - \frac{2GM}{c^2 r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) - c^2\left(1 - \tfrac{2GM}{c^2 r}\right)\mathrm{d}t^2 \tag{3.11}$$

(in Schwarzschild co-ordinates (r, θ, ϕ, t)).

This is only defined in the vacuum outside the body, but we can smoothly join it onto a different solution in the region containing matter. There is an apparent singularity at $r = r_S = \frac{2GM}{c^2}$, the Schwarzschild radius.

Usually the radius of matter is much greater than the Schwarzschild radius (for the Sun, $r_S = 3\mathrm{km}$), but if there is vacuum down to $r = r_S$ we have a black hole.

It can be proven that spherical symmetry and the Einstein equations imply the static Schwarzschild solution (even allowing time dependence). This is called Birkhoff's theorem.

Direct computation shows that ${R^a}_{bdc}$ has non-zero components and so this space-time is genuinely curved. The space part has metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1-\frac{2GM}{c^2r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

and is also curved.

As expected the corrections to the flat space metric are $\mathcal{O}(\frac{\phi}{c^2})$ in the far field.

3.3 Gravitational redshift in the Schwarzschild metric

The proper frequency as measured by the emitter is $b_1 = \frac{2\pi}{\mathrm{d}\tau_1} = \frac{2\pi c}{p_1(-g_{tt}(r_1))^{\frac{1}{2}}}$. The proper frequency measured by the reciever is $b_2 = \frac{2\pi}{\mathrm{d}\tau_2} = \frac{2\pi c}{p_1(-g_{tt}(r_2))^{\frac{1}{2}}}$.

The ratio

$$\frac{b_2}{b_1} = \sqrt{\frac{g_{tt}(r_1)}{g_{tt}(r_2)}} = \left(1-\frac{2GM}{c^2r_1}\right)^{\frac{1}{2}}\left(1-\frac{2GM}{c^2r_2}\right)^{-\frac{1}{2}}$$

gives the gravitational redshift. This is observed for many white dwarf stars.

3.4 Particle paths in the Schwarzschild metric

In this section we use geometrical units in which $c = G = 1$. We can obtain the geodesics from the Lagrangian $\mathcal{L}$ in (3.2). We do not attempt to solve the geodesic equations directly but instead seek first integrals of the motion.

$\frac{\partial L}{\partial \phi} = 0$ and so $r^2\sin^2\theta\dot{\phi} = h$, a constant. For a massive particle ($\lambda = \tau$) in the far field ($\tau \approx t$), we see that this is just the angular momentum (per unit mass) about the $\theta = 0$ axis.

$\frac{\partial L}{\partial t} = 0$, so that $\left(1-\frac{2M}{r}\right)\dot{t} = E$ is a constant. This is the energy per unit mass.

For a slow moving massive particle, $\frac{\mathrm{d}t}{\mathrm{d}\tau} \approx \left(1-v^2\right)^{-\frac{1}{2}}\left(1+\frac{M}{r}\right) \approx 1 + \frac{M}{r} + \frac{1}{2}v^2$. Thus in the far field, for a slow moving massive particle we see that $E \approx 1 + \frac{1}{2}v^2 - \frac{M}{r}$, and so it is reasonable to associate E with the energy.

Finally, as particle paths are geodesics,

$$\frac{\mathrm{D}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda} = 0$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\right) = 0.$$

Thus $\mathcal{L}$ is conserved. In fact, for a timelike geodesic $\mathcal{L} = -1$ and for a spacelike geodesic $\mathcal{L} = 0$.

We can further simplify the problem by taking the motion only in the equatorial plane. We can initially arrange $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$ by rotating the co-ordinates. Note that (3.4) is now automatically satisfied.

Using the conserved quantities we get the radial equation

$$-\frac{E^2}{1-\frac{2M}{r}} + \frac{\dot{r}^2}{1-\frac{2M}{r}} + \frac{h^2}{r^2} = \begin{cases} -1 & \text{massive particle} \\ 0 & \text{massless particle.} \end{cases}$$

In principle we can integrate this to get $\lambda(r)$, $\phi(r)$ and $t(r)$. For spatial orbits we use $u = r^{-1}$, so that $\dot{t} = -h\frac{\mathrm{d}u}{\mathrm{d}\phi}$.

For a massive particle the radial equation becomes

$$h^2\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = E^2 - (1+h^2u^2)(1-2Mu).$$

Taking $\frac{\mathrm{d}}{\mathrm{d}\phi}$ of this and dividing by $\frac{\mathrm{d}u}{\mathrm{d}\phi}$ we get

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{M}{h^2} + 3Mu^2.$$

The massless version of this is

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = 3Mu^2.$$

3.5 Perihelion advance

Consider bound orbits of a slow massive particle at large r ($r \gg M$). We seek to solve the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{M}{h^2} + 3Mu^2,$$

which we will do by perturbation methods. The zeroth approximation is

$$u = \frac{1+e\cos\phi}{l}, \qquad l = \frac{h^2}{m}.$$

We iterate this: the next approximation to $u(\phi)$ satisfies

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{1}{l} + \frac{3h^2}{l^3}\left(1 + \tfrac{1}{2}e^2 + 2e\cos\phi + \tfrac{1}{2}e^2\cos 2\phi\right).$$

It will be a better approximation if $h \ll l$. The solution of this equation is

$$lu = 1 + \tfrac{3h^2}{l^2}(1+\tfrac{1}{2}e^2) + \frac{h^2e}{l}\left(3\phi\sin\phi - \tfrac{1}{2}e\cos 2\phi\right) + e\cos\phi.$$

The $e\cos\phi$ comes from the zeroth order solution. The aperiodic $\phi\sin\phi$ term corresponds to an altered periodicity. Note that

$$e\cos\phi + \tfrac{3h^2e}{l^2}\phi\sin\phi \approx e\cos\left(1 - \tfrac{3h^2}{l^2}\right)\phi,$$

and the periodicity in ϕ is approximately $2\pi\left(1+\frac{3h^2}{l^2}\right)$. If a is the semi-major axis with

$$2a = \frac{1}{u_{\min}} + \frac{1}{u_{\max}} = \frac{2l}{1-e^2}$$

we can write the perihelion advance

$$\frac{6\pi h^2}{l^2} = \frac{6\pi M}{l} = \frac{6\pi M}{a(1-e^2)},$$

which is $\frac{6\pi GM}{c^2 a(1-e^2)}$ in MKS units.

The orbit is approximately elliptical, but is slowly rotating (precessing). The perihelion advance is $\frac{6\pi M}{a(1-e^2)}$ per orbit.

In the solar system the largest effect is on Mercury — the residual precession (that not accounted for by n-body Newtonian effects) of $43''$ per century measured agrees with this calculated result.

In a binary pulsar this effect is much bigger — about 4° per year.

3.6 Light deflection

Consider a particle on a null geodesic, satisfying the equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = 3Mu^2.$$

The zeroth order approximation for the light path is $u = \frac{\sin\phi}{R}$. The next approximation is

$$u = \frac{\sin\phi + \frac{M}{2R}(3+\cos 2\phi)}{R},$$

keeping the symmetry about $\phi = \frac{\pi}{2}$.

The light path is bent and we need to find ϵ. We set $u = 0$ and $\sin\epsilon \approx \epsilon$, $\cos 2\epsilon \approx 1$, so that $\epsilon \approx \frac{2M}{R}$. This is $\frac{2GM}{c^2 R}$ in MKS units.

This is observed in the solar system for light from stars which passes close to the sun at eclipses.

More detailed analysis of light deflection shows that the $\mathcal{O}(\frac{\phi}{c^2})$ corrections in g_{ij} and g_{tt} produce comparable contributions. In other theories of matter they combine differently, for instance Nordström's theory predicts no light deflection.

3.7 Black holes and the event horizon

Consider the vacuum Schwarzschild metric near $r = 2M$ and look at particles/photons falling towards $r = 2M$. For radial infall we have (θ, ϕ) constant and $h = 0$. We want to solve the equations

$$\left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\lambda} = E$$

and

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 - E^2 = \begin{cases} -\left(1 - \frac{2M}{r}\right) & \text{massive} \\ 0 & \text{massless.} \end{cases}$$

In the massive case $\lambda = \tau$ and so

$$\mathrm{d}\tau = -\frac{\mathrm{d}r}{\left(E^2 - 1 + \frac{2M}{r}\right)^{\frac{1}{2}}}.$$

We clearly need $E^2 > 1$. We see that $r \to 2M$ in a finite proper time

$$\tau = -\int \frac{\mathrm{d}r}{\left(E^2 - 1 + \frac{2M}{r}\right)^{\frac{1}{2}}}.$$

The co-ordinate time is nastier. We have

$$\mathrm{d}t = -\frac{E\mathrm{d}r}{(1 - \frac{2M}{r})\left(E^2 - 1 + \frac{2M}{r}\right)^{\frac{1}{2}}}$$

and so $t \to \infty$ as $r \to 2M$. Something similar happens for photons. We conclude that t is not a good co-ordinate for analysing the metric near $r = 2M$. Instead we use a co-ordinate tied to the incoming particles. It is simplest to do photons.

Consider radially infalling photons, which satisfy the equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\left(1 - \frac{2M}{r}\right).$$

We can integrate this to find $t = -r - 2M\log(r - 2M) + v$, where v is constant on photon paths. We change co-ordinates from (r, θ, ϕ, t) to (r, θ, ϕ, v), *ingoing Eddington-Finkelstein co-ordinates*. The Schwarzschild metric becomes

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right).$$

This metric is well-behaved down to $r = 0$ (except for a trivial polar co-ordinate singularity at $\theta = (0, \pi)$). It has $\det g_{ab} < 0$ and canonical form $+{+}{+}-$ everywhere in $v > 0$. It provides the ingoing extension of the Schwarzschild metric through $r = 2M$. This is a simple *co-ordinate singularity*.

However we find the curvature invariant $R_{abdc}R^{abdc} = \frac{48M^2}{r^6}$ and so there is a genuine singularity of the spacetime at $r = 0$. This is a *curvature singularity* and constitutes a boundary of the spacetime.

On any worldline we need $\mathrm{d}s^2 \leq 0$ (equality iff photons). Thus

$$-\left(1-\frac{2M}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r \leq 0$$

with equality iff we have photons with $\mathrm{d}\theta = \mathrm{d}\phi = 0$. In $r > 2M$ the future light cone is defined by

$$\mathrm{d}v \geq \frac{2\mathrm{d}r}{1-\frac{2M}{r}} \qquad \mathrm{d}v \geq 0.$$

In $r < 2M$ we have

$$\mathrm{d}r \leq \left(1-\frac{2M}{r}\right)\mathrm{d}v \qquad \mathrm{d}v \geq 0$$

and so $\mathrm{d}r \leq 0$. Thus any particle in $r < 2M$ inevitably has r decreasing to zero. Light or particles cannot escape from $r < 2M$, but *can* clearly escape from $r > 2M$. The region $r < 2M$ is a *black hole* and the boundary surface $r = 2M$ is its *event horizon*.

Chapter 4

Matter in General Relativity

Our final aim is to formulate the nongravitational laws of physics in curved spacetime and to find the field equations of GR in the presence of matter.

4.1 Physical laws

The equivalence principle means that all laws have their usual special relativistic forms in any LIFs. Moreover, the formulation of the laws should be the same in any reference frame — tensorial. Therefore to find physical laws we take the special relativistic laws and use them at the centre of a LIF to find the curved space covariant law.

In a LIF at P, $g_{ab} = \eta_{ab}$, $g_{ab,d} = 0$ and $\Gamma^a_{bd} = 0$ and so covariant derivatives reduce to partial derivatives. Therefore to make a special relativistic law covariant we replace partial derivatives with covariant derivatives and η_{ab} with g_{ab}. This is *minimal coupling* — we do not make unnecessary changes to the flat space laws.

For instance, consider free particle motion, which satisfies the equation $\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} = 0$ in a local inertial frame. This becomes

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a_{bd}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\frac{\mathrm{d}x^d}{\mathrm{d}\tau} = \frac{\mathrm{D}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\right) = 0,$$

which is the geodesic law.

A scalar field ψ satisfying the wave equation $\Box\psi = \eta^{ab}\psi_{,ab} = 0$ in flat spacetime becomes $g^{ab}\psi_{;ab} = 0$ in curved spacetime.

4.2 Energy-momentum tensors

The matter content of spacetime is described by an energy-momentum tensor T^{ab}.

Consider a continuous medium of density ρ, without pressure ("dust"). ρ is the proper density measured in the local inertial rest frame. Let $T^{ab} = \rho u^a u^b = T^{ba}$. In a local inertial frame we have $T^{ab}{}_{,b} = 0$ (by Navier-Stokes and the continuity equation) and so the equations of motion in general co-ordinates are $T^{ab}{}_{;b} = 0$. In the Newtonian limit, with gravity, the space parts of this give Navier-Stokes and the time part gives the continuity equation.

All forms of matter have symmetric energy-momentum tensors T^{ab} obeying $T^{ab}{}_{;b}$. This is ultimately because all quantum fields have a Lagrangian from which one can construct an energy-momentum tensor which is automatically conserved.

4.3 The Einstein field equations

We wish to generalise the vacuum Einstein equations $R_{ab} = 0$ to include matter sources and reproduce $\phi_{,ii} = 4\pi G\rho$ in the Newtonian limit.

4.3.1 The Bianchi identities

In a local inertial frame centred at $x^a = 0$ we have $g_{ab} = \eta_{ab} +$ quadratic and $\Gamma^a_{bd} =$ linear in a Taylor expansion about $x^a = 0$. Then

$$R^a{}_{bdc} = \Gamma^a_{bc,d} - \Gamma^a_{bd,c} + \text{quadratic}$$

and so

$$R^a{}_{bdc;e} = \Gamma^a_{bc,de} - \Gamma^a_{bd,ce} + \text{quadratic.}$$

Hence at the origin of a local inertial frame, $R^a{}_{b[dc;e]} = \Gamma^a_{b[c,de]} - \Gamma^a_{b[d,ce]} = 0$. But this is a tensorial equation, so $R^a{}_{b[dc;\xi]} = 0$ everywhere. These are the *Bianchi identities*.

They can be equivalently written (using the symmetries of $R^a{}_{bdc}$) as

$$R^a{}_{bdc;e} + R^a{}_{bce;d} + R^a{}_{bed;c} = 0.$$

contracting on a and c, multiplying by g^{be} and renaming the indices gives

$$-R^a{}_{b;a} + R_{;b} - R^a{}_{b;a} = 0.$$

We thus obtain the *contracted Bianchi identities*,

$$\nabla_b \left(R^{ab} - \tfrac{1}{2}g^{ab}R\right) = 0$$

4.3.2 Field equations

The contracted Bianchi identities suggest taking the field equations

$$R^{ab} - \tfrac{1}{2}Rg^{ab} = \kappa T^{ab}.$$

These are the *Einstein field equations*. The Bianchi identities then imply the conservation of energy-momentum automatically. In fact it can be shown that the left hand side of the Einstein equations is the only possible tensorial expression linear in $g_{ab,dc}$, not involving higher derivatives, vanishing in flat spacetime and with identically zero divergence.

One can verify the Newtonian limit: it turns out that $\kappa = \frac{8\pi G}{c^4}$ (Einstein's constant of gravitation).

Gravitation is nonlinear. The gravitational field must carry energy, although this can never be localised, since the geometry near any point P looks Minkowskian in a local inertial frame at P.

References

- Landau and Lifschitz, *The Classical Theory of Fields*, Fourth ed., Butterworth-Heinemann, 1975.

  The more I read this book the better I think it is. The style of presentation is very different from that used in this course with least action principles being used whenever possible. This style is more to my liking but you may have a different opinion.

- *Electrodynamics*, unpublished, 1997.

  Another field theory which the interested reader may like to take a look at. It was presented in a more quickfire manner than this course.