THE CLASSICAL GROUPS AS A SOURCE OF ALGEBRAIC PROBLEMS

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Introduction. Rather than attempting a general survey of what has been accomplished in algebra, I shall try to describe the influence on the development of algebra of one of the great sources of algebraic problems and ideas—the classical linear groups. I intend to show that these groups have provided an experimental, empirical basis for large parts of algebra. They are a family of examples which serve as raw material from which problems, conjectures, abstract ideas and general theorems have emerged, and which have provided direction and continuity to the research based on them. Surveys of recent developments and problems of current interest have been given by Brauer [5], [6], Carter [11], Dieudonné [24], M. Hall [26], Kaplansky [35], and MacLane [36].

It is impossible in this lecture to give an account of all the major results which involve the classical groups. In making a selection, I have simply included the material that has been most helpful to me.

1. Definitions and background. The story begins with Galois, who proved that a polynomial equation was solvable by radicals if and only if a certain finite group of automorphisms of a field was solvable. Thus he raised the problem of investigating groups in terms of their structure, and observed, as an example, that the set of all 2 by 2 invertible matrices with coefficients in a finite field of more than 3 elements, was a finite group that was not solvable.

Let us first give an up-to-date version of Galois' example. Let K be an arbitrary field, and V a finite dimensional vector space over K. The set of all linear transformations T of V which possess multiplicative inverses (and which we shall call invertible transformations) forms a group with respect to multiplication, called the general linear group $GL(V)$ of V. Hermann Weyl invented the term *the classical groups* to stand for $GL(V)$ and certain subgroups of $GL(V)$, which include the following ones.

(a) $SL(V)$, the unimodular group, consisting of all invertible transformations of determinant one.

(b) $O(V, f)$, the orthogonal group of a non-degenerate symmetric bilinear form f on V, consisting of all invertible transformations T that leave f invariant:

$$f(Tx, Ty) = f(x, y),\ x, y \in V.$$

For example, if V is n-dimensional euclidean space and $f(x, y)$ the usual inner product, $O(V, f)$ is the familiar group of linear transformations T that preserve lengths: $\|Tx\| = \|x\|$, where $\|x\| = f(x, x)^{1/2}$.

(c) $Sp(V, F)$, the symplectic group, consisting of all invertible transformations leaving invariant a non-degenerate skew-symmetric $(F(x, y) = -F(y, x))$ bilinear form F on V.

In order to understand why the list of classical groups does not continue, we have to return to Galois' observation about the structure of the classical groups.

Suppose we have, more generally, a family of algebraic systems $\{G_\alpha\}$ for which the fundamental theorem of homomorphisms is valid, such as groups, rings, algebras, etc. Whenever we have a homomorphism

$$\phi: G \xrightarrow[\text{onto}]{} H$$

with kernel L, G is called an extension of H with kernel L. G is called *simple* if whenever G is an extension of H with kernel L, then either G is isomorphic to H, or G is isomorphic to L. Given a family $\{G_\alpha\}$ of simple systems, the *extension problem* asks for the construction of all systems G for which there is a sequence of homomorphisms

$$\phi_1: G \xrightarrow[\text{onto}]{} G_{\alpha_1} \in \{G_\alpha\}$$

with kernel G_1,

$$\phi_2: G_1 \xrightarrow[\text{onto}]{} G_{\alpha_2} \in \{G_\alpha\}$$

with kernel G_2, etc., and such that for some k, the kernel G_k belongs to the family $\{G_\alpha\}$. The systems G determined in this way are built up from the simple systems $\{G_\alpha\}$ by solving the extension problem.

Solvable groups are the result of applying this process to the family $\{G_\alpha\}$ of cyclic groups of prime order. Galois' observation that the classical groups are not solvable shows that the classical groups are built up from non-abelian simple groups by solutions of the extension problem.

The problem of which simple groups are derived from the classical groups was investigated first by Jordan and Dickson ([34], [21]) with Dickson's book in 1901 containing fairly complete results for the classical groups defined over finite fields. Dickson's proofs, while satisfactory at the time, involved exhausting calculations with matrices which are a tribute to his persistence, but often left obscure the essential ideas. The subject was taken up again by Dieudonné and Artin ([22], [23], [3]) who identified the simple groups derived from the classical groups over arbitrary fields by elegant methods using the geometric properties of linear transformations and subspaces of the underlying vector space.

The result of their investigations, which extended and clarified Dickson's work, can be summarized as follows. First of all consider all homomorphisms f_α of a group G onto abelian groups A_α with kernels G_α. It turns out that there is a unique minimal such kernel G' called the *commutator group* of G, which is generated by all commutators $(x, y) = xyx^{-1}y^{-1}$, $x, y, \in G$. Thus an arbitrary group is an extension of an abelian group with kernel equal to the commutator group. The first step in investigating the structure of classical groups was to find the commutator groups. In the case of $GL(V)$ the multiplication theorem for determinants gives a homomorphism of $GL(V)$ onto the abelian multiplicative group of the field. With one exception (when V is the two dimensional space over the field of two elements), the commutator group turns out simply to be the kernel of the determinant homomorphism, $SL(V)$. The symplectic group is its own commutator group. The commutator group of the orthogonal group is more complicated. When isotropic vectors $v \neq 0$ such that $f(v, v)=0$ are present, the commutator group of the orthogonal group is the kernel of a homomorphism called the spinor norm, whose construction depends on a certain associative algebra called the Clifford algebra of the form f, (see [23], [3]).

The main structure theorem on the classical groups, proved in special cases by Dickson, and in full generality by Dieudonné, is that, with a manageable number of exceptions, the commutator group G' of a classical group G is an extension of a non-abelian simple group with kernel Center(G'). In other words $G'/\text{Center}(G')$ is a simple group, in most cases.

Dickson accumulated many other facts about the classical groups that proved to be useful for experimental purposes: he found the orders of the finite simple groups he had obtained, and partial information on their conjugacy classes and subgroups. All these results provided evidence that there should be general methods for studying these groups, without considering them one family at a time, as is the case in the approach both of Dickson and Dieudonné-Artin.

2. First attempts at a classification; Lie algebras. With the examples of simple groups furnished by the classical groups, and the alternating and Mathieu groups in the finite case, the problem arises to classify the non-abelian simple groups in some reasonable way, and in particular, to ask whether there are others. This problem, for arbitrary groups, still seems well beyond our reach, but is basic because the simple groups are the building blocks of arbitrary groups.

The first successful classification of simple algebraic systems which are relevant to our problem was contained in E. Cartan's thesis [9] in 1894, in which the simple Lie algebras over the field of complex numbers were classified. Since then Lie algebras have developed a life of their own (see [38], [32], [35]). The link between Cartan's work and group theory, however, can only be understood by plunging into the deep waters of Lie group theory, invented by Cartan's predecessor, Sophus Lie. For an up-to-date account of Lie group theory see Chevalley [13], Helgason [27], or Hochschild [29]. Thus the first successful classification of some kind of simple groups required heavy use of nonalgebraic methods, and shows that it may be shortsighted to expect a problem stated in algebraic terms to have a purely algebraic solution.

Without giving details, a complex Lie group G is a group which is at the same time a complex analytic manifold, such that the group operations are given locally by analytic functions. The classical matrix groups, with the operation of matrix multiplication, are the prototypes of Lie groups. From the manifold structure, a Lie group G has a tangent space $\mathfrak{g}$ at the identity, which is a finite dimensional vector space over the complex numbers. It is possible to define on $\mathfrak{g}$

a bilinear multiplication $[X, Y]$, satisfying the axioms of a Lie algebra:

$$[XX] = 0, \qquad X \in \mathfrak{g}$$
$$[[XY]Z] + [[YZ]X] + [[ZX]Y] = 0, X, Y, Z \in \mathfrak{g}.$$

It can then be shown, by an interesting combination of algebraic and analytical reasoning, that the group multiplication in a certain neighborhood of the identity is completely determined by the multiplication in the Lie algebra. Simple Lie groups have simple Lie algebras (where the concept of a simple Lie algebra is defined by the general remarks on the extension problem in section 1). Then the purely algebraic problem of classifying the simple Lie algebras implies a local classification of simple Lie groups.

The classical groups over the field of complex numbers are Lie groups, and to the simple groups derived from the classical groups correspond simple Lie algebras. Cartan's great achievement was to prove, completing a program already begun by Killing, that besides the simple Lie algebras corresponding to the classical groups, there are only 5 additional simple ones, called the exceptional Lie algebras, of dimensions 14, 52, 78, 133, and 248. There is a simple Lie group belonging to each of the exceptional simple Lie algebras, and the actual construction of the exceptional algebras and groups, has been one of the most striking applications of the theory of alternative and Jordan algebras. (See e.g. [**30**], [**31**], [**32**], [**16**], [**47**].)

The importance of Cartan's classification is that questions about properties of simple Lie algebras and Lie groups can be answered by checking against each type of algebra (or group) in the classification. Many interesting general problems and conjectures have been made in this experimental way, and later general solutions, independent of the classification, have been found for those that survived the case-by-case examination.

One difficulty in studying Lie groups via Lie algebras is that different Lie groups can have the same Lie algebra, and although homomorphisms of Lie groups always generate homomorphisms of the Lie algebra, the reverse statement is not necessarily true. But the most fundamental criticism from the point of view of algebra is that the classical groups are defined for arbitrary fields, while the Lie group machinery works smoothly only for groups which are real or complex analytic manifolds.

3. Representation theory. The next milestone after E. Cartan's work was Hermann Weyl's series of papers in 1925–1926 [**50**] on the representation theory of simple Lie algebras and Lie groups.

There were two problems, partly suggested by physical applications, that motivated the work on the representations.

One arises from the observation that a classical group G on a vector space V admits a homomorphism into the general linear group of the space of tensors

$$V^{(m)} = V \otimes \cdots \otimes V, \ (m \text{ factors})$$

given by

$$A \to \underbrace{A \otimes \cdots \otimes A}_{m} = A^{(m)}.$$

The problem was to decompose the tensor space $V^{(m)}$ into minimal invariant subspaces relative to the set of transformations $A^{(m)}$, $A \in GL(V)$.

A second problem, also suggested by physics, was to split up the space of continuous functions on the 2-sphere into minimal invariant subspaces relative to the action of the rotation group in 3-dimensional space.

These problems are both special cases of the following general problem. A group G is said to have a representation on a vector space M over a field K if there is a homomorphism $T: G \to GL(M)$. A subspace N is called a G-subspace if $T(g)N \subset N$ for all $g \in G$. The representation on M is called *irreducible* if the only G-subspaces are the trivial subspace $\{0\}$ and the whole space M, and *completely reducible* if M is a direct sum of irreducible G-subspaces.

At the time of Weyl's papers, there was already an extensive theory of representations of finite groups, due mainly to Frobenius, Schur and Burnside. In particular it was known that every representation of a finite group in a field of characteristic zero was completely reducible. (See [20], p. 41. The bibliography of [20] contains references to the original papers on representations of finite groups.)

Weyl, completing work begun by Cartan, classified explicitly the irreducible representations of the simple Lie algebras over the complex field, and proved also the theorem of complete reducibility for representations of simple Lie algebras, by using integration over a compact group associated with the Lie algebra. The Lie group—Lie algebra correspondence then solves the corresponding problems for representations of Lie groups. The failure of homomorphisms of the Lie algebra to generate homomorphisms of the group is strikingly illustrated by the spin representation of the rotation group. In this case, a representation of the orthogonal Lie algebra gives rise to a representation of a two-sheeted covering group of the rotation group, which cannot be viewed as a representation of the rotation group itself ([7], [13], Chapter 2, Section XI).

Weyl's work led first to a purely algebraic proof of complete reducibility of representations of simple Lie algebras by van der Waerden and the physicist Casimir [12]. Then J. H. C. Whitehead gave a new and profoundly original proof of the theorem on complete reducibility ([52], [53]) that can be viewed as the beginning of the cohomology theory of Lie algebras, and contains the motivation for at least a chapter of homological algebra ([38], [10]).

In 1939, Weyl returned to the subject, and in his book on the classical groups [51], he gave a new determination of the irreducible tensor representations of the classical groups over fields of characteristic zero, using the Wedderburn structure theory of associative algebras in an ingenious manner due to Schur, Brauer and himself. This method avoided the Lie algebras and their accompanying analysis, and gave one of the first solutions since Dickson's book of a deep problem involving the classical groups by purely algebraic methods.

Weyl's paper ([50] p. 358–359) also contained an explicit formula for the characters of the irreducible representations of the simple Lie groups, which has inspired other combinatorial investigations on the representations of Lie groups and Lie algebras by Freudenthal, Kostant, Steinberg and others (see [32], Chapter VIII, for a complete account). This work includes a generalization of the Clebsch-Gordan formula for splitting up the tensor product of two irreducible representations of the unimodular group on a two dimensional space, into irreducible components (see [49], pp. 127–131).

4. Algebraic groups. The key to the discovery of a uniform approach to the classical groups was contained in Weyl's 1925–26 papers, where he proved (see [50] pp. 338–342) that a simple Lie algebra over the complex numbers was determined up to isomorphism by a certain finite group of permutations of one-dimensional subspaces of the Lie algebra. Weyl proved that these groups were all isomorphic to finite groups generated by reflections in euclidean space E^n.

In 1934–35, Coxeter proved ([17], [18]) that a finite group was isomorphic to a group generated by reflections in E^n if and only if as an abstract group, it had generators $S_1, \cdots, S_n$ which satisfied the defining relations

$$S_i^2 = 1,\ 1 \leqq i \leqq n, \text{ and } (S_iS_j)^{n_{ij}} = 1,\ i \neq j.$$

He determined all such groups, and proved that the ones corresponding to simple Lie algebras were precisely those groups generated by reflections satisfying a certain crystallographic condition on the angles between the reflecting hyperplanes.

In 1941, Stiefel proved that the Weyl-Coxeter group of a Lie algebra could be constructed within a compact Lie group belonging to the Lie algebra [41]. Specifically, the Weyl-Coxeter group of G is isomorphic to $N(T)/T$ where T is a maximal torus in G, and $N(T)$ its normalizer. Using Lie theory, it followed that the structure of G was determined, at least locally, by the finite group $N(T)/T$.

The stage was now set for attempting to classify simple groups of Lie type over arbitrary fields, now that an internal key to their structure had been found. To replace the analytical machinery of Lie groups, algebraic geometry was used, since by that time the foundations of algebraic geometry over arbitrary ground fields had been laid by Zariski and Andre Weil.

Hermann Weyl had already observed ([51], p. 147) that the classical groups could be viewed as the intersection of an algebraic variety in the space $M_n(K)$ of all n by n matrices over K with the general linear matrix group $GL(n, K)$. Such a group is called an *algebraic group*, and again the prototypes are the classical groups. For example the unimodular group is defined by the polynomial relation

$$\det(X) - 1 = \sum \pm x_{i_1 1}x_{i_2 2} \cdots x_{i_n n} - 1 = 0.$$

In 1956–58, Chevalley succeeded in classifying all simple algebraic groups over algebraically closed fields of arbitrary characteristic [15]. He proved that such a group was determined up to isomorphism by a Weyl-Coxeter group exactly as in the case of compact Lie groups, and that the Weyl-Coxeter groups which appeared satisfied the crystallographic restriction. Since the groups with these Weyl-Coxeter groups were known, it followed that the only simple algebraic groups were the classical groups and the five types of exceptional groups. The proof of Chevalley's result makes heavy use of the techniques of algebraic geometry. It depends on some basic work of A. Borel on solvable algebraic groups.

5. Finite groups of Lie type. Until 1955 no simple finite groups were discovered that were not already known to Dickson. In his famous Tôhoku Journal paper [14] of 1955, Chevalley constructed a simple group associated with every simple Lie algebra over the complex field and every field K, with some exceptions when K is the field of 2 or 3 elements. His construction yielded finite simple groups corresponding to the exceptional Lie algebras, and gave a uniform method for investigating the groups, which is based on new structural properties of the groups.

Chevalley proved first that a simple Lie algebra $\mathfrak{g}$ over the complex numbers has a basis $X_1, \cdots, X_n$ such that

$$[X_iX_j] = \sum c_{ijk}X_k,$$

where the $\{c_{ijk}\}$ are integers. This multiplication table serves to define a Lie algebra $\mathfrak{g}_K$ over an arbitrary field K, since the structure constants $\{c_{ijk}\}$ can be viewed as elements of K. The Chevalley group G associated with the Lie algebra $\mathfrak{g}$ and the field K is defined to be a certain subgroup of the automorphism group of $\mathfrak{g}_K$. Chevalley then proved by a long argument, but one that broke with tradition by treating all the groups at once, that with a few exceptions, the groups G were all simple.

For a Lie algebra $\mathfrak{g}$ associated with one of the classical groups, the Chevalley group of $\mathfrak{g}$ and K coincides with the simple group derived from the corresponding classical group [37]. In case K is the complex field, the Chevalley groups over K yield a complete set of simple Lie groups with complex parameters. If K is algebraically closed, of arbitrary characteristic, then the Chevalley groups give a complete set of examples of simple algebraic groups defined over K, by the classification theorem for algebraic groups. If K is a finite field, then the Chevalley groups include the finite simple groups investigated by Dickson, Dieudonné and Artin, and those associated with the exceptional Lie algebras give infinite families of finite simple groups not isomorphic to those in Dickson's original list.

The proof of this last fact requires formulas for the orders of the Chevalley groups defined over finite fields. Chevalley [14] derived these formulas using some topological properties of the Lie group of the same type, and recently L. Solomon [40] has given another derivation of the formulas using some topological properties of the Weyl-Coxeter group of the Lie algebra. No purely algebraic derivation of the formulas is known. A knowledge of the formulas is important because of a number theoretical study by Artin ([1], [2]) of the orders of the known simple groups which, among other things, gives a method of showing that different Chevalley groups in general cannot be isomorphic.

6. Further results on finite groups. The excitement over Chevalley's paper had hardly died down before new infinite families of finite simple groups were discovered, by Hertzig, Ree, Steinberg, Suzuki, and Tits (see [11]). Some of these were defined by analogy with the classification of simple algebraic or Lie groups, where it was known that among the groups parametrized by fields which were not algebraically closed, certain twisted versions of the Chevalley groups could occur. All are obtained from a Chevalley group by the following sort of construction. Certain Chevalley groups G, defined for certain fields K, admit automorphisms of period 2 or 3 such that the set of elements in G left fixed under the automorphism contains a new simple group. Of course we do not know whether we have seen the last of these constructions, especially in view of a new simple group defined by Janko [33], of order $11(11+1)(11^3-1)$, which is a subgroup of a Chevalley group defined over the field of 11 elements, but is not obtained by the preceding general method.

Nevertheless the situation has crystallized enough for us at least to hope for uniform description of all the known finite simple non-abelian groups. For this we are indebted above all to J. Tits, whose work is still in the process of publication. From a close inspection of the construction of the Chevalley groups, Tits developed the following very simple but still rather mysterious set of axioms which are satisfied by all the known finite simple groups except possibly the Janko group, and are suggested by the structural properties of the groups derived in Chevalley's Tôhoku Journal paper. (See [44], [45], [46].)

A group G (not necessarily finite) is said to have a BN-pair if G is generated by subgroups B and N such that

$$B \cap N \text{ is a normal subgroup of } N; \text{ let } H = B \cap N.$$

$$N/H = \langle w_1, \cdots, w_n \rangle, \; w_i^2 = 1,$$

and for all cosets $w \in N/H$, and generators w_i of N/H,

$$w_i B w \subset BwB \cup Bw_i wB$$

and for each i

$$w_i B w_i \neq B.$$

The group $W = N/H$ is called the Weyl group of G, for the reason that the group N/H of an arbitrary finite group with a BN-pair is isomorphic to a group generated by reflections in euclidean space. Abstract groups with BN-pairs can be expressed as a union of (B, B) double cosets, $G = \bigcup_{n \in N} BnB$, with the double cosets in 1–1 correspondence with the elements of the Weyl group, a decomposition discovered first for the classical groups by Bruhat [8]. All the Chevalley groups, as well as the twisted types, have BN-pairs. Among other things Tits has found an elegant proof of simplicity for certain types of groups with BN-pairs [46], that provides in almost all cases a much shorter and easier proof of simplicity of the Chevalley groups and the twisted types than the one given in Chevalley's paper.

It has also been possible, using the methods of Chevalley and Tits in combination with Lie algebra methods similar to those in Weyl's 1925–26 papers, to classify all the irreducible representations of the finite Chevalley groups and the twisted types, in an algebraically closed field of the same characteristic as the field by which the groups are parametrized, completing the work begun by Weyl for the classical groups over fields of characteristic zero ([42], [19]).

The complex representations and characters of the finite Chevalley groups have not yet been classified, partly because of our lack of knowledge about the conjugacy classes.

In this connection, the Jordan normal form theorem tells us that two elements X and Y of $GL(V)$ are conjugate, i.e., $X=ZYZ^{-1}$, if and only if they have the same Jordan normal form. The Bruhat decomposition $G=\cup BnB$ fails to give a normal form up to conjugacy for elements of a general Chevalley group. R. Steinberg [43] has recently given a normal form for certain conjugacy classes of elements in "simply connected" versions of all the Chevalley groups over algebraically closed fields, that includes the Jordan normal form as a special case. This powerful result has already had applications to the classification of algebraic groups over non-algebraically closed fields and may suggest a method of determining the conjugacy classes in finite Chevalley groups.

Behind all the fascinating special results and problems suggested by these developments, lies the austere and immensely difficult problem of classifying the finite simple groups. The classical groups on 2- and 3-dimensional spaces were determined in terms of the structure of their subgroups by Zassenhaus, Suzuki, and Brauer and his students. See Section 92 of [20] for a survey of this work up to 1962. Then came the great achievement for which the Cole prize in algebra was awarded at the Denver meeting of the American Mathematical Society last winter—Walter Feit and John Thompson's proof, published in 1963, of Burnside's conjecture that a non-abelian simple group must have even order [25]. Since then the pace in finite groups has picked up, with Thompson's proof that the only finite simple groups, all of whose proper subgroups are solvable, are the known ones, and related works of Suzuki and Gorenstein and Walter. This work brings together all that is known about finite groups because the inductive method of proof used in these questions presents one with a group whose structure is to be determined, given fairly complete information on the structure of its subgroups which may be solvable groups, p-groups, etc.

If we continue to believe, as of course we must, in the possibility of communicating mathematical proofs as well as statements of results, then we are faced with the difficult and important task of simplifying some of these Herculean arguments, which often run to two and three hundred pages, to the point where they can meaningfully be presented in lectures and text books.

A bright ray of light in this direction is Tits' classification of finite groups with BN-pairs. It was known for some time that in a BN-pair G whose Weyl group was the symmetric group S_3, a coset geometry could be defined that satisfied the axioms for a projective plane. D. G. Higman and McLaughlin [28] proved that if the group G was finite, the plane was Desarguesian, and that G was an extension of the simple classical group $PSL(3, K)$ for some finite field K. Tits has proved that there is a geometry associated with every finite group with a BN-pair, and has succeeded in classifying the possible geometries in most cases [45]. The difficulty in applying his work to arbitrary finite groups is that it seems to be very difficult to tell whether a group has a BN-pair or not.

7. Conclusion. There are many other important directions I have not been able to take up. One basic experimental fact we have observed is the strong dependence of the methods used for studying the simple groups and their representations on the field over which the groups are defined. To make precise and formalize this kind of dependence is one of the tasks of homological algebra. In this direction we have, for example, applications of Galois cohomology [39] to the study of algebraic groups defined over nonalgebraically closed fields, and important new constructions in the category of modules over a ring—the algebraic K-theory of Bass [4]—which are at least partly suggested by the study of linear groups over the ring.

I have been able, nevertheless, to tell enough of the story to show that the classical groups have successfully resisted the best efforts of three generations of mathematicians to subdue them, and continue to suggest interesting problems for further research. The methods that have been used represent strong improvements and refinements over those available to Jordan and Dickson and have found their way into our elementary teaching. Their continual interaction with other branches of mathematics, and physics, would have heartened Hermann Weyl, who expressed in the preface of his book [51] his concern over the dangers of a too thorough specialization of mathematical research. The permanent feature of all this work has been the source of the problems—the groups themselves.

I hope that what I have said makes it clear that in our capacity as teachers we should introduce our students to the empirical sources, the physical facts of our science, as well as to the general ideas and methods we use in our work.

Other sources of algebraic ideas that have had a similar influence are the theory of algebraic numbers, representations of finite groups, and algebraic geometry; but these I leave to another time and other speakers.

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