

Physics of the neutrino mass

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New Journal of Physics **6** (2004) 82

Received 17 May 2004

Published 14 July 2004

Online at <http://www.njp.org/>

doi:10.1088/1367-2630/6/1/082

Abstract. Recent neutrino oscillation experiments have yielded valuable information on the nature of neutrino masses and mixings and qualify as the first evidence for physics beyond the standard model. Even though we are far from a complete understanding of the new physics implied by them, there are many useful hints. As the next precision era in neutrino physics is about to be launched, we review the physics of neutrino mass: what we have learned and what we are going to learn.

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Section 1

1.1. Introduction

For a long time, it was believed that neutrinos were massless, spin-half particles, making them drastically different from their other standard model spin-half cousins such as the charged leptons (e, μ, τ) and the quarks (u, d, s, c, t, b), which are known to have mass. In fact, the masslessness of the neutrino was considered so sacred in the 1950s and 1960s that the fundamental law of weak interaction physics, the successful V-A theory for charged current weak processes proposed by Sudarshan, Marshak, Feynman and Gell-Mann, was considered to be intimately linked to this fact. The argument went as follows: a massless fermion field equation is invariant under the γ_5 transformation; since the neutrino is one such particle and it participates exclusively in weak interactions, the weak interactions must somehow reflect invariance under the γ_5 -transformation of all fermionic matter (i.e. quarks, charged leptons and the neutrinos) participating in weak interactions. The argument is obviously very heuristic, but it is not hard to see its profound implication: it leads to the resulting four-Fermi weak interaction to involve only V-A currents. This argument remained persuasive for a long time since there was no evidence for neutrino mass for almost the next 50 years and became a celebrated myth in particle theory.

This myth has however been shattered by the accumulating evidence for neutrino mass from the solar and atmospheric neutrino data compiled in the 1990s and still ongoing. One must therefore now be free to look beyond the γ_5 invariance idea for exploring new physics as we proceed to understand the neutrino mass.

The possibility of a nonzero neutrino mass at phenomenological level goes back almost 50 years.¹ In the context of gauge theories, they were discussed extensively in the 1970s and 1980s long before there was any firm evidence for it. For instance, the left-right symmetric theories of weak interactions introduced in 1974 and discussed in those days in connection with the structure of neutral current weak interactions, predicted nonzero neutrino mass as a necessary consequence of parity invariance and quark lepton symmetry.

The existence of a nonzero neutrino mass makes neutrinos more like the quarks, and allows for mixing between the different neutrino species leading to the phenomenon of neutrino oscillation, an idea first discussed by Pontecorvo [2] and Maki *et al* [2] in the 1960s, unleashing a whole new realm of particle physics phenomena to explore. More importantly, the simple fact that neutrino masses vanish in the standard model implies that evidence for neutrino mass is solid evidence for the existence of new physics beyond the standard model.

We are of course far from a complete picture of the masses and mixings of the various neutrinos and cannot therefore have a full outline of the theory of neutrino masses at present.

¹ For a summary of the recent developments in neutrino physics as well as the historical developments in the field, see [1].

However, there exist enough information and indirect indications that constrain the masses and mixings among the neutrinos that we can see a narrowing of the possibilities for the theories beyond the standard model. Combined with other ideas outside the neutrino arena such as supersymmetry and unification, the possibility narrows even further. Many clever experiments now in progress will soon clarify or rule out many of the allowed models. It will be one of the goals of this paper to give a panoramic view of the most likely scenarios for new physics that explain what is now known about neutrino masses.² Such discussions are of course by nature very subjective and therefore a sincere apology is due at the beginning of this discussion to all those whose ideas are not cited.

We hope to emphasize two kinds of ideas: one which provides a general framework for understanding of the small neutrino masses, the seesaw mechanism and predicts the existence of superheavy right-handed neutrinos. In the opinion of this author, these ideas are likely to be part of the final theory of neutrino masses. We then touch briefly on some specific models that are based on the above general framework but attempt to provide an understanding of the detailed mass and mixing patterns. These works are instructive for several reasons: first they provide proof of the detailed workability of the general ideas described above (sort of existence proofs that things will work); secondly they often illustrate the kind of assumptions needed and through that provide a unique insight into which directions the next step should be; finally of course nature may be generous in picking one of those models as the final message bearer.

In discussing the neutrino mass, it is instructive to compare it with the other well-known fermion, the electron. The electron and the neutrino are in many ways very similar particles: they are both spin-half objects; they both participate in weak interactions with same strength; in fact they are so similar that in the limit of exact gauge symmetries they are two states of the same object and therefore in principle indistinguishable. Yet there are profound differences between them in the standard model: after gauge symmetry breaking, only the electron has electric charge but the neutrinos come out electrically neutral as they should to match observations. Another difference is that only the left-handed neutrinos of each generation are included in the standard model and not their right-handed counterparts, whereas for the e , μ , τ and all the quark flavours, both helicity states are included. The fact that the right-handed neutrino is excluded from the standard model coupled with the fact that $B - L$ is an exact symmetry of the model implies that neutrino remains massless to all orders in perturbation theory as well as nonperturbatively, as we will discuss later on in this review.

The fact that the neutrino has no electric charge endows it with certain properties not shared by other fermions of the standard model. One can write two kinds of Lorentz-invariant mass terms for the neutrino, the Dirac and Majorana masses, whereas for the charged fermions, conservation of electric charge allows only Dirac-type mass terms. In the four-component notation for the fermions, the Dirac mass has the form $\bar{\psi}\psi$, whereas the Majorana mass is of the form $\psi^T C^{-1} \psi$, where ψ is the four-component spinor and C is the charge conjugation matrix. One can also discuss the two different kinds of mass terms using the two-component notation for the spinors, which provides a very useful way to discuss neutrino masses. We therefore present some of the salient concepts behind the two-component description of the neutrino.

² See [3] for a summary of theoretical and phenomenological ideas relating to massive neutrinos.

1.2. Two-component notation for neutrinos

Before we start the discussion of the two-component neutrino, let us write down the Dirac equation for an electron:³

$$i\gamma^\lambda \partial_\lambda \psi - m\psi = 0. \quad (1)$$

This equation follows from a free Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\lambda \partial_\lambda \psi - m\bar{\psi}\psi \quad (2)$$

and leads to the relativistic energy momentum relation $p^\lambda p_\lambda = m^2$ for the spin-half particle only if the four γ_λ s anticommute. If we take γ_λ s to be $n \times n$ matrices, the smallest value of n for which four anticommuting matrices exist is four. Therefore, ψ must be a four-component spinor. The physical meaning of the four components is as follows: two components for particle spin-up and spin-down and same for the antiparticle.

A spin-half particle is said to be a Majorana particle if the spinor field ψ satisfies the condition of being self-charge conjugate, i.e.

$$\psi = \psi^c \equiv C\bar{\psi}^T, \quad (3)$$

where C is the charge conjugation matrix and has the property $C\gamma_\lambda C^{-1} = -\gamma_\lambda^T$. This constraint reduces the number of independent components of the spinor by a factor of 2, since the particle and the antiparticle are now the same particle. Using this condition, the mass term in the Lagrangian in equation (2) can be written as $\psi^T C^{-1} \psi$, where we have used the fact that C is a unitary matrix. Writing the mass term in this way makes it clear that if a field carries a $U(1)$ charge and the theory is invariant under those $U(1)$ transformations, then the mass term is forbidden. This means that one cannot impose the Majorana condition on a particle that has a gauge charge. Since the neutrinos do not have electric charge, they can be Majorana particles unlike the quarks, electron or the muon. It is of course well known that the gauge boson interactions in a gauge theory Lagrangian conserve a global $U(1)$ symmetry known as lepton number with the neutrino and electron carrying the same lepton number. If lepton number were to be established as an exact symmetry of nature, the Majorana mass for the neutrino would be forbidden and the neutrino, like the electron, would be a Dirac particle.

The properties of a Majorana fermion can be seen in its free-field expansion in terms of creation and annihilation operators:

$$\psi(x) = \int \frac{d^3 p}{\sqrt{(2\pi)^3 2E_p}} \Sigma_s (a_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ip \cdot x} + a_s^\dagger v_s(\mathbf{p}) e^{ip \cdot x}). \quad (4)$$

In the γ matrix convention where

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad \text{and} \quad \gamma_0 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix},$$

the u_s and v_s are given by

$$u_s(\mathbf{p}) = \frac{m}{\sqrt{E}} \begin{pmatrix} \alpha_s \\ \frac{E - \sigma \cdot \mathbf{p}}{m} \alpha_s \end{pmatrix} \quad (5)$$

³ We follow the discussion in [4].

and

$$v_s(\mathbf{p}) = \frac{m}{\sqrt{E}} \begin{pmatrix} -\frac{E+\sigma\cdot\mathbf{p}}{m}\alpha'_s \\ \alpha'_s \end{pmatrix} \quad (6)$$

α_s and α'_s are two component spinors.

If we choose $\alpha'_s = \sigma_2 \alpha_s$, we get the relation among the spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ $C\gamma_0 u_s^*(\mathbf{p}) = v_s(\mathbf{p})$ and the Majorana condition follows. Note that if ψ were to describe a Dirac spinor, then we would have had a different creation operator b^\dagger in the second term in the free-field expansion above.

The origin of the two-component neutrino is rooted in the isomorphism between the Lorentz group and the $SL(2, C)$ group. The latter is defined as the set of 2×2 complex matrices with unit determinant, whose generators satisfy the same Lie algebra as that of the Lorentz group. Its basic representations are 2- and 2*-dimensional. These are the spinor representations and can be used to describe spin-half particles.

We can therefore write the familiar four-component Dirac spinor used in the text books to describe an electron can be written as

$$\psi = \begin{pmatrix} \phi \\ i\sigma_2 \chi^* \end{pmatrix},$$

where χ and ϕ two two-component spinors. A Dirac mass is given by $\chi^T \sigma_2 \phi$ whereas a Majorana mass is given by $\chi^T \sigma_2 \chi$, where σ_a are the Pauli matrices. To make correspondence with the four-component notation, we point out that ϕ and $i\sigma_2 \chi^*$ are nothing but the ψ_L and ψ_R , respectively. It is then clear that χ and ϕ have opposite electric charges; therefore the Dirac mass $\chi^T \sigma_2 \phi$ maintains electric charge conservation (as well as any other kind of charge like lepton number, etc).

Two-component neutrino is described by the following Lagrangian:

$$\mathcal{L} = v^\dagger i\sigma^\lambda \partial_\lambda v - \frac{im}{2} e^{i\delta} v^T \sigma_2 v + \frac{im}{2} e^{-i\delta} v^\dagger \sigma_2 v^*. \quad (7)$$

This leads to the following equation of motion for the field χ :

$$i\sigma^\lambda \partial_\lambda \chi - im\sigma_2 \chi^* = 0. \quad (8)$$

As is conventionally done in field theories, we can now give a free-field expansion of the two-component Majorana field in terms of the creation and annihilation operators:

$$\chi(x, t) = \sum_{p,s} [a_{\mathbf{p},s} \alpha_{p,s} e^{-ip\cdot x} + a_{\mathbf{p},s}^\dagger \beta_{p,s} e^{ip\cdot x}], \quad (9)$$

where the sum on s goes over the spin-up and spin-down states.

Using the field equations for a free massive two-component Majorana spinor, one can show that its expansion in terms of the creation and annihilation operators and two-component spinors

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is given by the following expression:

$$\begin{aligned} \chi(x, t) = & \sum_p [a_{p,+} e^{-ip \cdot x} - a_{p,-}^\dagger e^{ip \cdot x}] \alpha \sqrt{E+p} \\ & + \sum_p [a_{p,-} e^{-p \cdot x} + a_{p,+}^\dagger e^{ip \cdot x}] \beta \sqrt{E-p}. \end{aligned} \quad (10)$$

Note that in a beta decay process, where a neutron is annihilated and proton is created, the leptonic weak current that is involved is $\bar{e}v$ (dropping γ matrices); therefore, along with the electron, what is created predominantly is a right-handed particle (with a wave function α), the amplitude being of order $\sqrt{E+p} \approx \sqrt{2E}$. This is the right-handed antineutrino. The left-handed neutrino is produced with a much smaller amplitude $\sqrt{E-p} \approx m_n u/E$. Similarly, in the fusion reaction in the core of the Sun, what is produced is a left-handed state of the neutrino with a very tiny, i.e. $O(m_\nu/E)$ admixture of the right-handed helicity.

1.2.1. Neutrinoless double beta decay and neutrino mass. As already noted, a Majorana neutrino breaks lepton number by two units. This has the experimentally testable prediction that it leads to the process of neutrinoless double beta decay, where a nucleus (generally even–even nuclei) $(Z, N) \rightarrow (Z+2, N-2) + 2e^-$. We will now show by using the above property of the Majorana neutrino that if light neutrino exchange is responsible for this process, then the amplitude is proportional to the neutrino mass.

Double beta decay involves the change of two neutrons to two protons and therefore has to be a second-order weak interaction process. Since each weak interaction process emits an antineutrino, in second-order weak interaction, the final state will involve two anti-neutrinos. But in neutrinoless double beta decay, there are no neutrinos in the final state; therefore the two neutrinos must go into the vacuum state. The vacuum state by definition has no spin whereas the antineutrino emitted in a beta decay has spin. Consider the antineutrino from one of the decays: it must be predominantly right-handed. But to disappear into vacuum, it must combine with a left-handed antineutrino so that the left- and right-handed spin projections add up to zero. In the previous section, we showed that the fraction of left-handed spin projection in a neutrino emitted in beta decay is m_ν/E . Therefore, $\bar{\nu}_e \bar{\nu}_e \rightarrow |0\rangle$ must be proportional to the neutrino mass. Thus neutrinoless double beta decay is a very sensitive measure of neutrino mass.

1.2.2. Neutrino mass in two-component notation. Let us now discuss the general neutrino mass for Majorana neutrinos. We have seen earlier that for a Majorana neutrinos, there are two different ways to write a mass term consistent with relativistic invariance. This richness in the possibility for neutrino masses also has a down side in the sense that, in general, there are more parameters describing the masses of the neutrinos than those for the quarks and leptons. For instance, for the electron and quarks, dynamics (electric charge conservation) reduces the number of parameters in their mass matrix. As an example, using the two-component notation for all fermions, for the case of two two-component spinors, a charged fermion mass will be described only by one parameter whereas for a neutrino, there will be three parameters. This difference increases rapidly, e.g. for $2N$ spinors, to describe charged fermion masses, we need N^2 parameters (ignoring CP violation), whereas for neutrinos, we need $2N(2N+1)/2$ parameters. What is more interesting is that for a neutrino-like particle, one can have both even and odd number of two-component objects and have a consistent theory.

In this paper, we will use two-component notation for neutrinos. Thus when we say that there are N neutrinos, we will mean N two-component neutrinos.

In the two-component language, all massive neutrinos are Majorana particles and what is conventionally called a Dirac neutrino is really a very specific choice of mass parameters for the Majorana neutrino. Let us give some examples: if there is only one two-component neutrino (we will drop the prefix two-component henceforth), it can have a mass $m\nu^T\sigma_2\nu$ (to be called $\equiv m\nu\nu$ in short-handed notation). The neutrino is now a self-conjugate object which can be seen if we write an equivalent four-component spinor ψ :

$$\psi = \begin{pmatrix} \nu \\ i\sigma_2\nu^* \end{pmatrix}. \quad (11)$$

Note that this four-component spinor satisfies the condition

$$\psi = \psi^c \equiv C\bar{\psi}^T. \quad (12)$$

This condition implies that the neutrino is its own anti-particle, a fact more transparent in the four- rather than the two-component notation. The above exercise illustrates an important point, i.e. given any two-component spinor, one can always write a self-conjugate (or Majorana) four-component spinor. Whether a particle is really its own antiparticle or not is therefore determined by its interactions. To see this for the electrons, one may solve the following exercise: i.e. if we wrote two Majorana spinors using the two two-component spinors that describe the charged fermion (electron), then until we turn on the electromagnetic interactions and the mass term, we will not know whether the electron is its own antiparticle or not. Once we turn on the electromagnetism, this ambiguity is resolved.

Let us now go one step further and consider two two-component neutrinos (ν_1, ν_2). The general mass matrix for this case is given by

$$\mathcal{M}_{2\times 2} = \begin{pmatrix} m_1 & m_3 \\ m_3 & m_2 \end{pmatrix}. \quad (13)$$

Note first that this is a symmetric matrix and can be diagonalized by orthogonal transformations. The eigenstates which will be certain admixtures of the original neutrinos now describe self-conjugate particles. One can look at some special cases.

Case (i): If we have $m_{1,2} = 0$ and $m_3 \neq 0$, then one can assign a charge $+1$ to ν_1 and -1 to ν_2 under some $U(1)$ symmetry other than electromagnetism and the theory is invariant under this extra $U(1)$ symmetry, which can be identified as the lepton number and the particle is then called a Dirac neutrino. The point to be noted is that the Dirac neutrino is a special case of two Majorana neutrinos. In fact, if we insisted on calling this case one with two Majorana neutrinos, then the two will have equal and opposite (in sign) mass as can be seen diagonalizing the above mass matrix. Thus, a Dirac neutrino can be thought of as two Majorana neutrinos with equal and opposite (in sign) masses. Since the argument of a complex mass term in general refers to its C transformation property (i.e. $\psi^c = e^{i\delta_m}\psi$, where δ_m is the phase of the complex mass term), the two two-component fields of a Dirac neutrino have opposite charge conjugation properties.

Case (ii): If we have $m_{1,2} \ll m_3$, this case is called pseudo-Dirac neutrino since this is a slight departure from case (i). In reality, in this case also the neutrinos are Majorana neutrinos with their masses $\pm m_0 + \delta$ with $\delta \ll m_0$. The two-component neutrinos will be maximally mixed.

Thus this case is of great current physical interest in view of the atmospheric (and perhaps solar) neutrino data.

Case (iii): There is a third case where one may have $m_1 = 0$ and $m_3 \ll m_2$. In this case, the eigenvalues of the neutrino mass matrix are given, respectively, by $m_\nu \simeq -m_3^2/m_2$ and $M \simeq m_2$. One may wonder under what conditions such a situation may arise in a realistic gauge model. It turns out that if ν_1 transforms as an $SU(2)_L$ doublet and ν_2 is an $SU(2)_L$ singlet, then the value of m_3 is limited by the weak scale, whereas m_2 has no such limit and $m_1 = 0$ if the theory has no $SU(2)_L$ triplet field (as for instance is the case in the standard model). Choosing $m_2 \gg m_3$ then provides a natural way to understand the smallness of the neutrino masses. This is known as the seesaw mechanism [5]. Since this case is very different from cases (i) and (ii), it is generally said that in grand unified theories, one expects the neutrinos to be Majorana particles. The reason is that in most grand unified theories there is a higher scale which under appropriate situations provides a natural home for the large mass m_2 .

While we have so far used only two neutrinos to exemplify the various cases including the seesaw mechanism, these discussions generalize when $m_{1,2,3}$ are each $N \times N$ matrices (which we denote by $M_{1,2,3}$). For example, the seesaw formula for this general situation can be written as

$$\mathcal{M}_\nu \simeq -M_{3D}^T M_{2R}^{-1} M_{3D}, \quad (14)$$

where the subscripts D and R are used in anticipation of their origin in gauge theories where M_D turns out to be the Dirac matrix and M_R is the mass matrix of the right-handed neutrinos and all eigenvalues of M_R are much larger than the elements of M_D . It is also worth pointing out that equation (13) can be written in a more general form, where the Dirac matrices are not necessarily square matrices but $N \times M$ matrices with $N \neq M$. We give such examples below.

Although there is no experimental proof that neutrino is a Majorana particle, the general opinion is that since the seesaw mechanism provides such a simple way to understand the glaring differences between the masses of the neutrinos and the charged fermions and since it implies that the neutrinos are Majorana fermions, they indeed are most likely to be Majorana particles.

Even though, for most situations, the neutrino can be treated as a two-component object regardless of whether its mass is of Dirac or Majorana type, there are certain practical situations where differences between the Majorana and Dirac neutrino become explicit: one case is when the two neutrinos annihilate. For Dirac neutrinos, the particle and the antiparticle are distinct and therefore their annihilation is not restricted by Pauli principle in any manner. However, for the case of Majorana neutrinos, the identity of neutrinos and antineutrinos plays an important role and one finds that the annihilation to the Z-bosons occurs only via the P-waves. Similarly, in the decay of the neutrino to any final state, the decay rate for the Majorana neutrino is a factor of 2 higher than for the Dirac neutrino.

1.3. Experimental indications for neutrino masses

Much has been said about the experimental evidences for neutrino masses and their analyses to determine the current favourite values for the various mass differences as well as mixing angles. I will therefore only summarize the main points that are relevant for our present understanding of neutrino masses and for the sake of completeness (see [6] for a detailed discussion).

At present there is no conclusive evidence for neutrino masses from direct search experiments for neutrino masses using tritium beta decay and neutrinoless double beta decay (see later).

There are however many experiments that have measured the flux of neutrinos from the Sun and the cosmic rays and which have provided clear evidences for neutrino oscillations i.e. neutrinos of one flavour transmuting to neutrinos of another flavour. Since such transmutation can occur only if the neutrinos have masses and mixings, these experiments provide evidence for neutrino mass. The expression for vacuum oscillation probability for neutrinos of a given energy E that have travelled a distance L is given by

$$P_{\alpha\beta} = \sum_{i,j} |U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}| \cos \left(\frac{\Delta_{ij} L}{2E} - \phi_{\alpha\beta,ij} \right). \quad (15)$$

From this it is clear that neutrino oscillation data yield information about the mass difference squares of the neutrinos ($\Delta_{ij} = m_i^2 - m_j^2$) and mixing angles $U_{\alpha i}$. If the neutrino propagates in dense matter, the Mikheyev–Smirnov–Wolfenstein effect [8] will change this formula but it also depends on the same parameters mass difference square and the $U_{\alpha i}$ [9]. The analysis of the data for the atmospheric neutrinos where the neutrino propagates in vacuum and that for solar neutrinos where the effect of dense matter in the Sun is important therefore leads to the following picture for masses and mixings.

1.3.1. Atmospheric neutrinos. In the Super-Kamiokande experiment [10] which confirms the indications of oscillations in earlier data from the Kamiokande, IMB experiments. More recent data from Soudan II and MACRO experiments provide further confirmation of this evidence. The observation here is the following: in the standard model with massless neutrinos, all the muon and electron neutrinos produced at the top of the atmosphere would be expected to reach detectors on the Earth and would be isotropic; what has been observed is that while that is true for the electron neutrinos, the muon neutrino flux observed on the Earth exhibits a strong zenith angle dependence. A simple way to understand this would be to assume that the muon neutrinos oscillate into another undetected species of neutrino on their way to the earth, with a characteristic oscillation length of order of 10 000 km. Since the oscillation length is roughly given by $E(\text{GeV})/\Delta m^2(\text{eV}^2)$ km, for a GeV neutrino, one would expect the particle physics parameter Δm^2 corresponding to the mass difference between the two neutrinos to be around 10^{-33} eV² corresponding to maximal mixing.

From the existing data several important conclusions can be drawn: (i) the data cannot be fit assuming oscillation between ν_μ and ν_e nor $\nu_\mu - \nu_s$, where ν_s is a sterile neutrino which does not any direct weak interaction; and (ii) the oscillation scenario that fit the data best is $\nu_\mu - \nu_\tau$ for the mass and mixing parameters

$$\Delta m_{\nu_\mu - \nu_\tau}^2 \simeq (2-8) \times 10^{-3} \text{eV}^2, \quad \sin^2 2\theta_{\mu-\tau} \simeq 0.8-1. \quad (16)$$

1.3.2. Solar neutrinos. The second evidence for neutrino oscillation comes from the seven experiments that have observed a deficit in the flux of neutrinos from the Sun as compared with the predictions of the standard solar model championed by Bahcall and his collaborators [6] and more recently studied by many groups. The experiments responsible for this discovery are the Chlorine, Kamiokande, Gallex, SAGE, Super-Kamiokande, SNO, GNO [7] experiments conducted at the Homestake mine, Kamioka in Japan, Gran Sasso in Italy and Baksan in Russia and Sudbery in Canada. The different experiments see different parts of the solar neutrino spectrum. The details of these considerations are discussed in other lectures. The oscillation interpretation of the solar

neutrino deficit has more facets to it than the atmospheric case: first the final state particle that the ν_e oscillates into and second what kind of Δm^2 and mixings fit the data. At the moment there is a multitude of possibilities. Let us summarize them now.

As far as the final state goes, it can either be one of the two remaining active neutrinos, ν_μ and ν_τ or it can be the sterile neutrino ν_s . SNO neutral current data announced recently has very strongly constrained the second possibility (i.e. the sterile neutrino in the final state). The global analyses of all solar neutrino data seem to favour the so-called large mixing angle MSW solution with parameters $\Delta m^2 \simeq 1.2 \times 10^{-5} - 3.1 \times 10^{-4} \text{ eV}^2$; $\sin^2 2\theta \simeq 0.58 - 0.95$.

1.3.3. LSND. Finally, we come to the last indication of neutrino oscillation from the Los Alamos Liquid Scintillation Detector (LSND) experiment [15], where neutrino oscillations both from a stopped muon (DAR) as well as the one accompanying the muon in pion decay (known as the DIF) have been observed. The evidence from the DAR is statistically more significant and is an oscillation from $\bar{\nu}_\mu$ to $\bar{\nu}_e$. The mass and mixing parameter range that fits data is

$$\text{LSND: } \Delta m^2 \simeq 0.2 - 2 \text{ eV}^2, \quad \sin^2 2\theta \simeq 0.003 - 0.03. \quad (17)$$

There are also points at higher masses specifically at 6 eV^2 , which are also allowed by the present LSND data for small mixings. KARMEN experiment at the Rutherford laboratory has very strongly constrained the allowed parameter range of the LSND data [16]. Currently, the Miniboone experiment at Fermilab is in progress to probe the LSND parameter region [17].

1.3.4. Neutrinoless double beta decay and tritium decay experiment. Oscillation experiments only depend on the difference of mass squares of the different neutrinos and the mixing angles. Therefore, to have a complete picture of neutrino masses, we need other experiments. Two such experiments are the neutrinoless double beta decay searches and the search for neutrino mass from the analysis of the end point of the electron energy spectrum in tritium beta decay.

Neutrinoless double beta decay measures the following combination of masses and mixing angles:

$$\langle m \rangle_{\beta\beta} = \sum_i U_{ei}^2 m_i. \quad (18)$$

Therefore, naively speaking, it is sensitive to the overall neutrino mass scale. However, in practice, as we will see below, for the case of both normal and inverted hierarchies, it is most likely to settle the question of the overall mass scale at the presently contemplated level of sensitivity in double beta decay searches. Only if the neutrino mass patterns are inverse hierarchical does one expect a visible signal in $\beta\beta_{0\nu}$ decay. We do not go into great detail into this issue, except to mention that in drawing any conclusions about neutrino mass from this process, one has to first have a good calculation of nuclear matrix elements of the various nuclei involved such as ^{76}Ge , ^{136}Xe , ^{100}Mo , etc; secondly, another confusing issue has to do with alternative physics contributions to $\beta\beta_{0\nu}$ which are unrelated to neutrino mass. Nevertheless, neutrinoless double beta decay is a fundamental experiment and a nonzero signal will establish a fundamental result that neutrino is a Majorana particle and that lepton number symmetry is violated. Regardless of whether it tells us anything about the neutrino masses, it would provide a fundamental new revelation about physics beyond the standard model. At present, two experiments, Heidelberg–Moscow and IGEX, that use enriched ^{76}Ge have published limits of $\leq 0.3 \text{ eV}$ [12].

More recently, evidence for a double beta signal in the Heidelberg–Moscow data has been claimed [13].

Another important result in further understanding of neutrino mass physics could come from the tritium end-point searches for neutrino masses. This experiment will measure the parameter $m_\nu = \sqrt{\sum_i |U_{ei}|^2 m_i^2}$. This involves a different combination of masses and mixing angles than $\langle m \rangle_{\beta\beta}$. At present, the KATRIN proposal for a high-sensitive search for m_ν has been made and it is expected that it can reach a sensitivity of 0.3 eV.

A third source of information on neutrino mass will come from cosmology, where more detailed study of structure in the universe is expected to provide an upper limit on $\sum_i m_i$ of less than an eV.

Our goal now is to study the theoretical implications of these discoveries. We will proceed towards this goal in the following manner: we will isolate the mass patterns that fit the above data and then look for plausible models that can first lead to the general feature that neutrinos have tiny masses; then we would try to understand in simple manner some of the features indicated by the data in the hope that these general ideas will be part of our final understanding of the neutrino masses. As mentioned earlier, to understand the neutrino masses one has to go beyond the standard model. First we will sharpen what we mean by this statement. Then we will present some ideas which may form the basic framework for constructing the detailed models.

1.4. Patterns and textures for neutrinos

As already mentioned, we will assume two-component neutrinos and therefore their masses will in general be Majorana type. Let us also give our notation to facilitate further discussion: the neutrinos emitted in weak processes such as the beta decay or muon decay are weak eigenstates and are not mass eigenstates. The mass eigenstates determine how a neutrino state evolves in time. Similarly, in the detection process, it is the weak eigenstate that is picked out. This is of course the key idea behind neutrino oscillation and the formula presented in the last section. To set the notation, let us express the weak eigenstates in terms of the mass eigenstates. We will denote the weak eigenstate by the symbol α, β or simply e, μ, τ etc whereas the mass eigenstate will be denoted by the symbols i, j, k , etc. To relate the weak eigenstates to the mass eigenstates, let us start with the mass terms in the Lagrangian for the neutrino and the charged leptons:

$$\mathcal{L}_m = \nu_L^\dagger \mathcal{M}_\nu \nu_L + \bar{E}_L M_\ell E_R + \text{h.c.} \quad (19)$$

Here ν and E , which denote the column vectors for neutrinos and charged leptons, are in the weak basis. To go to the mass basis, we diagonalize these matrices as follows:

$$U_L^\dagger \mathcal{M}_\nu U_L = d_\nu, \quad V_L M_\ell V_R^\dagger = d_\ell. \quad (20)$$

The physical neutrino mixing matrix is then given by

$$\mathbf{U} = V_L U_L. \quad (21)$$

$U_{\alpha i}$ and relate the two sets of eigenstates (weak and mass) as follows:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \quad (22)$$

Using this equation, one can derive the well-known oscillation formulae for the survival probability of a particular weak eigenstate α discussed in the previous section.

To see the general structure of the mixing matrix \mathbf{U} , let us recall that the matrix \mathcal{M}_ν is complex and symmetric and therefore has six complex parameters describing it for the case of three generations. However, since the neutrino is described by a complex field, we can redefine the phases of three fields to remove three parameters. That leaves nine parameters. In terms of observables, there are three mass eigenvalues (m_1, m_2, m_3) and three mixing angles and phases in the mixing matrix \mathbf{U} . The three phases can be split into one Dirac phase, which is analogous to the phase in the quark mixing matrix and two Majorana phases. We can then write the matrix \mathbf{U} as

$$\mathbf{U} = U^{(0)} \begin{pmatrix} 1 & & \\ & e^{i\phi_1} & \\ & & e^{i\phi_2} \end{pmatrix}. \quad (23)$$

The matrix $U^{(0)}$ has three real angles $\theta_{12}, \theta_{23}, \theta_{13}$ and a phase. The goal of experiments is to determine all nine of these parameters. The knowledge of the nine observables allows one to construct the mass matrix for the neutrinos and from there one can go in search of the new physics beyond the standard model that leads to such a mass matrix.

The neutrino mass observables given above can be separated into two classes: (i) oscillation observables and (ii) nonoscillation observables. The first class of observables are those accessible to neutrino oscillation experiments and are the two mass differences Δm_{\odot}^2 and Δm_A^2 ; three mixing angles θ_{12} (or θ_{\odot}), θ_{23} (or θ_A) and θ_{13} (the reactor angle, also called U_{e3}) and the CP phase δ in $U^{(0)}$. The remaining three observables which can only be probed by nonoscillation experiments are the lightest mass of the three neutrinos and the two Majorana phases $\phi_{1,2}$.

1.5. Neutrino mixing matrix and mass patterns

Let us first discuss to what extent the oscillation observables are known. Our discussion will focus on the three neutrino fits to all neutrino data which give as central value for $\Delta m_A^2 \simeq 0.0025 \text{ eV}^2$; for solar neutrinos, it gives $\Delta m_{\odot}^2 \simeq (2-20) \times 10^{-5} \text{ eV}^2$. It also provides information on the angles in \mathbf{U} which can be summarized by the following mixing matrix (neglecting all CP phases):

$$\mathbf{U} = \begin{pmatrix} c & s & \epsilon \\ -\frac{s+c\epsilon}{\sqrt{2}} & \frac{c-s\epsilon}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{s-c\epsilon}{\sqrt{2}} & \frac{-c-s\epsilon}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (24)$$

where $\epsilon \leq 0.16$ from the CHOOZ and PALO-VERDE reactor experiments [14]. s is the solar neutrino mixing angle. Present experiments allow the range $0.6 \leq \sin^2 2\theta_{\odot} \leq 0.96$ with the central being near 0.8. The atmospheric mixing angle θ_A is close to maximal i.e. $\sin^2 2\theta_A \simeq 0.8-1$.

As far as the mass pattern goes however, there are three possibilities:

- normal hierarchy: $m_1 \ll m_2 \ll m_3$,
- inverted hierarchy: $m_1 \simeq -m_2 \gg m_3$ and
- approximately degenerate pattern [18] $m_1 \simeq m_2 \simeq m_3$,

Table 1. Different possible conclusions regarding the nature of the neutrinos and their mass hierarchy from the three.

$\beta\beta_{0\nu}$	Δm_{13}^2	KATRIN	Conclusion
yes	> 0	yes	Degenerate, Majorana
yes	> 0	no and heavy particle contribution	Degenerate, Majorana
yes	< 0	no	Inverted, Majorana
yes	< 0	yes	Degenerate, Majorana
no	> 0	no	Normal, Dirac or Majorana
no	< 0	no	Dirac or Pseudo-Dirac
no	< 0	yes	Dirac or Pseudo-Dirac
no	> 0	yes	Dirac or Pseudo-Dirac

where m_i are the eigenvalues of the neutrino mass matrix. In the first case, the atmospheric and the solar neutrino data give direct information on m_3 and m_2 , respectively. On the other hand, in the last case, the mass differences between the first and the second eigenvalues will be chosen to fit the solar neutrino data and the second and the third to fit the atmospheric neutrino data.

Since Majorana masses violate lepton number, a very important constraint on any discussion of neutrino mass patterns arises from the negative searches for neutrinoless double beta decay [11]. The most stringent present limits are obtained from the Heidelberg–Moscow enriched Germanium-76 experiment at Gran Sasso and implies an upper limit on the following combination of masses and mixings:

$$\langle m_\nu \rangle \equiv \sum_i U_{ei}^2 m_{\nu_i} \leq 0.35 \text{ eV}, \quad 95\% \text{ c.l.} \quad (25)$$

This upper limit depends on the nuclear matrix element calculated by the Heidelberg group [11]. There could be an uncertainty of a factor of 2 in this estimate. This would then relax the above upper bound to at most 0.7 eV in the worst-case scenario. This is still a very useful limit and becomes especially relevant when one considers whether the neutrinos constitute a significant fraction of the hot dark matter of the universe. A useful working formula is $\sum_i m_{\nu_i} \simeq 24\Omega_\nu \text{ eV}$, where Ω_ν is the neutrino fraction that contributes to the dark matter of the universe. For instance, if the neutrino contribution to dark matter fraction is 20%, then the sum total of neutrino masses must be 4.8 eV. Such large values are apparently in disagreement with present upper limits from structure surveys such as the 2dF survey and others.

Neutrinoless double beta decay limits also imply very stringent constraints on the mixing pattern in the degenerate case.

The inverted hierarchy case (ii) is quite an interesting one and will be discussed from a theoretical perspective in more detail later on; but at the moment we simply note that in this case the value of m_1 is nothing but the $\sqrt{\Delta m_A^2}$. The solar mass difference is an additional parameter in the mass matrix.

In table 1, we describe what we can learn about the nature of the neutrino from the various experiments.

At this point, it is appropriate to stress the theoretical challenges raised by the existing neutrino oscillation data.

- *Ultralight neutrinos*: Why are the neutrino masses so much lighter than the quark and charged lepton masses?
- *Near bimaximal mixing*: How to understand simultaneously two large mixing angles one for the $\mu - \tau$ and another for $e - \mu$?
- *Smallness of $\Delta m_{\odot}^2 / \Delta m_A^2$* : Experimentally, $\Delta m_{\odot}^2 \simeq 10^{-2} \Delta m_A^2$. How does one understand this in a natural manner?
- *Smallness of U_{e3}* : The reactor results also seem to indicate that the angle $\theta_{13} \equiv U_{e3}$ is a very small number. One must also understand this in a framework that simultaneously explains all other puzzles.

Possible other puzzles include a proper understanding of neutrino mass degeneracy if there is a large positive signal for the neutrinoless double beta decay and of course, when we have evidence for CP violating phases in the mass matrix, we must understand their magnitude.

1.6. Neutrino mass textures

From the mixing matrix in equation (24), we can write down the allowed neutrino mass matrix for any arbitrary mass pattern assuming the neutrino is a Majorana fermion. Denoting the matrix elements of \mathcal{M}_ν as $\mu_{\alpha\beta}$ for $\alpha, \beta = 1, 2, 3$, we have (recall that $\mu_{\alpha\beta} = \mu_{\beta\alpha}$)

$$\begin{aligned}
 \mu_{11} &= [c^2 m_1 + s^2 m_2 + \epsilon^2 m_3], \\
 \mu_{12} &= (1/\sqrt{2})[-c(s + c\epsilon)m_1 + s(c - s\epsilon)m_2 + \epsilon m_3], \\
 \mu_{13} &= (1/\sqrt{2})[-c(s - c\epsilon)m_1 - s(c + s\epsilon)m_2 + \epsilon m_3], \\
 \mu_{22} &= \frac{1}{2}[(s + c\epsilon)^2 m_1 + (c - s\epsilon)^2 m_2 + m_3], \\
 \mu_{23} &= \frac{1}{2}[-(s^2 - c^2 \epsilon^2)m_1 - (c^2 - s^2 \epsilon^2)m_2 + m_3], \\
 \mu_{33} &= \frac{1}{2}[(s - c\epsilon)^2 m_1 + (c + s\epsilon)^2 m_2 + m_3].
 \end{aligned} \tag{26}$$

One can use equation (26) to get information on the nature of the neutrino mass matrix and use it to get clues to the nature of physics beyond the standard model. One technique is to rewrite this mass matrix in a way that reflects some new underlying symmetry of physics beyond the standard model. Below we give some examples of mass matrices that are closely related to this mass matrix but reflect some symmetries of the lepton world. A second utility of equation (26) is to search for mass matrices that lead to testable predictions and thereby test models.

As an example consider the case when $\mu_{11} = \mu_{22} = \mu_{33} = 0$. This is the prediction from the Zee model [19]. Using equation (26), one can easily see that in the leading order the vanishing of all diagonal entries implies that $\Delta m_{\text{dot}}^2 = 0$, i.e. one must keep higher-order terms in ϵ to get a nonzero Δm_{\odot}^2 .

An elementary way to proceed in this direction is to start with ways to understand the maximal mixing using only two flavours as in the case of the atmospheric neutrinos (i.e. the 2–3 sector of the neutrino flavour). It is well known that if we have a matrix of the form

$$M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \tag{27}$$

then its eigenstates are maximal admixtures of the original ‘flavour’ states. The eigenvalues are $(A + B)$, $(A - B)$. It is now clear that if we want one of the eigenvalues to be much less than the other, we must have $A \approx B$. We will now have to generalize this discussion to the case of three generations. For this we note that the ‘mixing matrix’ for this case can be written as

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (28)$$

We will call this matrix ‘maximal mixing matrix’.

Theorem. *The maximal mixing matrix for N generations, if N is a prime number, is given by*

$$U_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot \\ 1 & z & z^2 & z^3 & \cdot \\ 1 & z^2 & z^4 & \cdot & \cdot \\ 1 & z^3 & z^6 & \cdot & \cdot \\ 1 & z^4 & z^8 & \cdot & \cdot \end{pmatrix}, \quad (29)$$

where z is the N th root of unity and the rows and columns extend in an obvious manner to make the matrix $N \times N$.

Note that the general form of an arbitrary element is z^{pq} . When $pq \geq N$, the power is simply given by $pq - N$. When the number of generations is not a prime number, then $N = N_1 \cdot N_2 \cdot N_3 \cdots$. In this case,

$$U_N = U_{N_1} \times U_{N_2} \times U_{N_3} \times, \quad (30)$$

where \times stands for a direct product. For example for the case of $N = 4$, we have for the maximal mixing matrix $U_4 = U_2 \times U_2$ which is given by

$$U_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad (31)$$

The 4×4 case turns out to be interesting since by decoupling a linear combination of the states from the 4×4 system, we can get the exact bimaximal form [21]. The corresponding 3×3 mass matrix can also be obtained by starting from this discussion. To see how one gets that, note that the matrix in equation (24) for the 2×2 case has a S_{12} symmetry. The corresponding symmetry in the 4×4 case is then $S_{12} \times S_{34}$. Using this property and decoupling one linear combination of the fields, we can get the mass matrix for the 3×3 case that leads to bimaximal mixing [22]. It is left as an exercise to the reader to show that the corresponding mass matrix is given by [21]

$$M_\nu = \begin{pmatrix} A + D & F & F \\ F & A & D \\ F & D & A \end{pmatrix}. \quad (32)$$

Since the present data imply that there are deviations from the exact bimaximal form, this mass matrix must only be treated as a leading-order contribution.

The three different mass patterns can emerge from this mass matrix in various limits: e.g. (i) for $F \ll A \simeq -D$, one gets the normal hierarchy; (ii) for $F \gg A, D$, one has the inverted pattern for masses and (iii) the parameter region $F, D \ll A$ leads to the degenerate case. Clearly, this mass matrix is the leading-order matrix and there will have to be small corrections to this to fit solar neutrino data as well as any possible evidence (or lack of it) for neutrinoless double beta decay. An interesting symmetry of this mass matrix is the $\nu_\mu \leftrightarrow \nu_\tau$ interchange symmetry, which is obvious from the matrix; however in the limit where $A = D = 0$, there appears a much more interesting symmetry, i.e. the continuous symmetry $L_e - L_\mu - L_\tau$ [23] (see also [55]). If the inverted mass matrix is confirmed by future experiments, this symmetry will provide an important clue to new neutrino-related physics beyond the standard model (see e.g. [24]). There are other ways to proceed towards the same goal. One way inspired by the studies of quark mass matrices is to consider mass matrices with zeros in it and hope that there are sensible symmetries that will guarantee the zeros. One may then be able to obtain relations between different observables such as masses and mixing angles that can be tested in experiments.

To proceed towards this goal, let us recall that in the absence of CP violation, the neutrino mass matrix has six independent entries. In terms of observables there are five oscillation observables, two mass differences ($\Delta m_\odot^2, \Delta m_A^2$) and three mixing angles: solar, atmospheric and the reactor angle U_{e3} . So if we have more than one vanishing element, we will have nontrivial relations between oscillation observables which can be used as tests of the various ansatzes.

The first point to note is that if there are three zeros or more, the mass matrix cannot describe data. The proof of this is left as an exercise to the reader.

Let us therefore consider some typical two zero mass matrices [25]:

$$M_\nu = \sqrt{\Delta m_A^2} \begin{pmatrix} 0 & 0 & d\epsilon \\ 0 & 1 + a\epsilon & 1 \\ d\epsilon & 1 & 1 + b\epsilon \end{pmatrix} \quad (33)$$

or, alternatively,

$$M_\nu = \sqrt{\Delta m_A^2} \begin{pmatrix} 0 & d\epsilon & 0 \\ d\epsilon & 1 + a\epsilon & 1 \\ 0 & 1 & 1 + b\epsilon \end{pmatrix}. \quad (34)$$

They give a four-parameter fit to all data and yield a relation between $\Delta m_\odot^2/\Delta m_A^2$, U_{e3} , θ_\odot and θ_A [25]:

$$U_{e3}^2 \cos 2\theta_\odot = \frac{\sin^2 2\theta_\odot}{4} \frac{\Delta m_\odot^2}{\Delta m_A^2}. \quad (35)$$

This relation predicts that $U_{e3} \geq 0.12$ and can be tested in planned long baseline experiments to be conducted in the near future.

There are also some other two zero textures that predict a degenerate spectrum for neutrinos.

So far we have focused on studying the form of the neutrino mass matrix. The implicit assumption in these discussions has been that the charged lepton mass matrix is diagonal. It may however be that all neutrino mixings are a reflection of structure in the charged lepton mass matrix rather than in the neutrinos. An example of this kind is the democratic mass matrix [27], which also leads to a near bimaximal mixing. As was noted in [28], to get the democratic mixing

matrix, one must choose the charged lepton mass matrix in the following form while keeping the neutrino mass matrix diagonal:

$$M_\ell = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}. \quad (36)$$

This form can be derived from a permutation symmetry S_3 operating on the lepton doublets, which may provide another clue to possible gauge model building.

Other models that use charged lepton sector to understand the neutrino have been the focus of several papers [29], especially in the context of grand unified theories [29]. We do not discuss this class of models here.

1.7. Inverted hierarchy and $L_e-L_\mu-L_\tau$ symmetry

In this section, we consider another interesting clue to model building present in neutrino data if the mass arrangement is inverted. As already noted, if in equation (33), we set $A = D = 0$, this leads to a neutrino mass matrix with two degenerate neutrinos with mass $\pm\sqrt{2}F$ and one massless neutrino. The atmospheric mass difference is given by $\Delta m_A^2 = 2F^2$ and mixing angle $\theta_A = \pi/4$. As far as the solar ν_e oscillation is concerned, $\sin^2 2\theta_\odot = 1$ but $\Delta m_\odot^2 = 0$. While this is unphysical, this raises the hope that as corrections to this mass matrix are taken into account, it may be possible to understand the smallness of $\Delta m_\odot^2/\Delta m_A^2$ naturally.

In fact, this hope is fortified by the observation that the $A = D = 0$ limit of the mass matrix in [33] has the leptonic symmetry $L_e-L_\mu-L_\tau$; therefore one might hope that as this symmetry is broken by small terms, one will end up with a situation that fits the data well.

This question was studied in [30, 72]. To proceed with the discussion, let us consider the following mass matrix for neutrinos where small $L_e-L_\mu-L_\tau$ violating terms have been added:

$$\mathcal{M}_\nu = m \begin{pmatrix} z & c & s \\ c & y & d \\ s & d & x \end{pmatrix}. \quad (37)$$

The charged lepton mass matrix is chosen to have a diagonal form in this basis and $L_e-L_\mu-L_\tau$ symmetric.

In the perturbative approximation, we find the following sum rules involving the neutrino observables and the elements of the neutrino mass matrix. The two obvious relations are

$$\sin^2 2\theta_A = \sin^2 2\theta + \mathcal{O}(\delta^2), \quad (38)$$

$$D_3 \equiv \Delta m_A^2 = -m^2 + 2\Delta m_\odot^2 + \mathcal{O}(\delta^2).$$

The nontrivial relations that also hold for this model are

$$\sin^2 2\theta_\odot = 1 - \left(\frac{\Delta m_\odot^2}{4\Delta m_A^2} - z \right)^2 + \mathcal{O}(\delta^3),$$

$$\frac{\Delta m_\odot^2}{\Delta m_A^2} = 2(z + \vec{v} \cdot \vec{x}) + \mathcal{O}(\delta^2), \quad (39)$$

$$U_{e3} = \vec{A} \cdot (\vec{v} \times \vec{x}) + \mathcal{O}(\delta^3),$$

where $\vec{v} = (\cos^2 \theta, \sin^2 \theta, \sqrt{2} \sin \theta \cos \theta)$, $\vec{x} = (x, y, \sqrt{2}d)$ and $\vec{A} = (1/\sqrt{2})(1, 1, 0)$. δ in the preceding equations represents the small parameters in the mass matrix. These equations represent one of the main results of this paper. Below we study their implications. Finally, there is the relation $\langle m \rangle_{\beta\beta} = mz$. This is an exact relation true to all orders in the small parameters.

One of the major consequences of these relations is that (i) there is a close connection between the measured value of the solar mixing angle and the neutrino mass measured in neutrinoless double beta decay; and (ii) the present values for the solar mixing angle can be used to predict the $m_{\beta\beta}$ for a value of the Δm_{\odot}^2 . For instance, for $\sin^2 2\theta_{\odot} = 0.9$, we would predict $(\Delta m_{\odot}^2/4\Delta m_A^2 - z) = 0.3$. For small Δm_{\odot}^2 , this implies $m_{\beta\beta} \simeq 0.01$ eV. The second relation involving the $\Delta m_{\odot}^2/\Delta m_A^2$ in terms of x, y, z, d tells us that for this to be the case, we must have strong cancellation between the various small parameters. Given this, the above $m_{\beta\beta}$ value is expected to be within the reach of new double beta decay experiments contemplated [32]. Note however that the $\sin^2 2\theta_{\odot}$ cannot be greater than 0.9 in the case of approximate $L_e-L_{\mu}-L_{\tau}$ symmetry.

If the value of $\sin^2 2\theta_{\odot}$ is ultimately determined to be less than 0.9, the question one may ask is whether the idea of $L_e-L_{\mu}-L_{\tau}$ symmetry is dead. The answer is in the negative since so far we have explored the breaking of $L_e-L_{\mu}-L_{\tau}$ symmetry only in the neutrino mass matrix. It was shown in [30] that if the symmetry is broken in the charged lepton mass, one can lower the $\sin^2 2\theta_{\odot}$ as long as the value of U_{e3} is sizeable. However, given the present upper limit on U_{e3} , the smallest value is somewhere around $\sin^2 2\theta_{\odot} \simeq 0.8$.

1.8. CP violation

A not very well-explored aspect of neutrino physics at the moment is CP violation in lepton physics. Unlike the quark sector, CP violation for Majorana neutrinos allows for more phases for neutrinos. Since the Majorana neutrino mass matrix is symmetric, for N generations of neutrinos, there are in general $N(N+1)/2$ phases in it. When the mass matrix is diagonalized, these phases will appear in the unitary matrix U_L that does the diagonalization (i.e. $U^T - \mathcal{M}_\nu U_L = d_\nu$). If we are working in a basis where the charged lepton mass matrix is diagonal, then U_L is a leptonic weak mixing matrix. As we have seen, this has $N(N+1)/2$ phases. Out of them, redefinition of the charged lepton fields in the weak current allows the removal of N phases; so there are $N(N-1)/2$ phases in the neutrino masses. In the quark sector, both up and down fields could be redefined allowing for the number of physical phases that appear in the end to be smaller. However, for Majorana neutrinos, redefinition of the fields does not remove the phases entirely from the theory but rather shifts them to other places where they can manifest themselves physically [33].

Thus, for two generations, there is one and for three generations there are three phases. A convenient parameterization of the mixing matrix with these phases in the mixing matrix for the case of three generations is **UK**:

$$\mathbf{U} = \begin{pmatrix} c & s & \epsilon e^{-i\delta} \\ -\frac{s + c\epsilon e^{i\delta}}{\sqrt{2}} & \frac{c - s\epsilon e^{i\delta}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{s - c\epsilon e^{i\delta}}{\sqrt{2}} & \frac{-c - s\epsilon e^{i\delta}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (40)$$

and

$$\mathbf{K} = \text{Diag}(1, e^{i\phi_1}, e^{i\phi_2}). \quad (41)$$

The phase δ is called the Dirac phase and the ϕ_i are called Majorana phases. Note that the Dirac phase is always multiplied by the small mixing angle ϵ . Its measurability is therefore very closely tied to the absolute magnitude of ϵ . Coming to the Majorana phases, one of the two phases ϕ_i can, in principle, be probed in neutrinoless double beta decay. To see this, let us note that

$$\langle m \rangle_{\beta\beta} = |m_1 c^2 + m_2 s^2 e^{i\phi_1} + m_3 \epsilon^2 e^{-i(\delta - \phi_2)}|. \quad (42)$$

Since $\epsilon \ll 1$, the last term in the above equation can be dropped. It is then easy to see that one has some chance of seeing the CP phase ϕ_1 , once one has a precise knowledge of the Δm_A^2 and the solar mixing angle, provided the nuclear matrix elements are known better [34]. More optimistically, when the phase is zero, the two terms in the expression for $\beta\beta_{0\nu}$ add up and the chances for seeing it is enhanced.

On the whole, there is some chance of measuring the CP phase both for the inverted and the degenerate mass case provided the nuclear matrix elements have much smaller uncertainties than presently known, whereas for the case of normal hierarchy, it depends on how small the smallest neutrino mass is. If it is very close to m_2 , the so-called quasi-degenerate case, then one has a good chance to measure the phase if the $\langle m \rangle_{\beta\beta}$ is measured to a precision of 0.001 eV.

A very interesting question relates to the observability of truly CP violating leptonic processes. This question has been addressed in [35]. For instance, although one can probe the CP phase in neutrinoless double beta decay, it is not really a CP violating process; in other words, even in the presence of CP violation, the rate for neutrinoless double beta decay of a nucleus is same as that for the corresponding anti-nucleus. On the other hand, there are genuine CP violating processes where the CP phase can be probed. One is the celebrated example of early universe leptogenesis where one looks at the decay of the Majorana right-handed neutrino into $\ell + H$ and $\bar{\ell} + \bar{H}$ and it is their difference that manifests as the lepton asymmetry. Similarly, one can look at rare decays of K^\pm to $\pi^\mp + \mu^\pm + \mu^\pm$ and similar decay modes for the B meson, where the presence of a physical intermediate state leads to the observability of a truly CP violating difference between decay rates. It however appears that truly CP violating neutrino processes are much too suppressed to be observable.

Section 2

2.1. Why neutrino mass requires physics beyond the standard model

We will now show that in the standard model, the neutrino mass vanishes to all orders in perturbation theory as well as nonperturbatively. The standard model is based on the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ under which the quarks and leptons transform as described in the table 2.

The electroweak symmetry $SU(2)_L \times U(1)_Y$ is broken by the vacuum expectation of the Higgs doublet $\langle H^0 \rangle = v_{wk} \simeq 246$ GeV, which gives mass to the gauge bosons and the fermions, all fermions except the neutrino. Thus, the neutrino is massless in the standard model, at the tree-level. There are several questions that arise at this stage. What happens when one goes beyond

Table 2. The assignment of particles to the standard model gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$.

Field	Gauge transformation
Quarks Q_L	$(3, 2, \frac{1}{3})$
Right-handed up-quarks u_R	$(3, 1, \frac{4}{3})$
Right-handed down-quarks d_R	$(3, 1, -\frac{2}{3})$
Left-handed leptons L	$(1, 2, -1)$
Right-handed leptons e_R	$(1, 1, -2)$
Higgs boson \mathbf{H}	$(1, 2, +1)$
Colour gauge fields G_a	$(8, 1, 0)$
Weak gauge fields W^\pm, Z, γ	$(1, 3, +1, 0)$

the above simple tree-level approximation? Secondly, do nonperturbative effects change this tree-level result? Finally, how can we judge how this result will be modified when the quantum gravity effects are included?

The first and second questions are easily answered by using the $B - L$ symmetry of the standard model. The point is that since the standard model has no $SU(2)_L$ singlet neutrino-like field, the only possible mass terms that are allowed by Lorentz invariance are of the form $v_{iL}^T C^{-1} v_{jL}$, where i, j stand for the generation index and C is the Lorentz charge conjugation matrix. Since the v_{iL} is part of the $SU(2)_L$ doublet field and has lepton number $+1$, the above neutrino mass term transforms as an $SU(2)_L$ triplet and, furthermore, it violates total lepton number (defined as $L \equiv L_e + L_\mu + L_\tau$) by two units. However, a quick look at the standard model Lagrangian convinces one that the model has exact lepton number symmetry after symmetry breaking; therefore such terms can never arise in perturbation theory. Thus, to all orders in perturbation theory, the neutrinos are massless. As far as the nonperturbative effects go, the only known source is the weak instanton effects. Such effects could affect the result if they broke the lepton number symmetry. One way to see if such breaking occurs is to look for anomalies in lepton number current conservation from triangle diagrams. Indeed, $\partial_\mu j_\ell^\mu = cW\tilde{W} + c'B\tilde{B}$ due to the contribution of the leptons to the triangle involving the lepton number current and W s or B s. Luckily, it turns out that the anomaly contribution to the baryon number current nonconservation has also an identical form, so that the $B - L$ current j_{B-L}^μ is conserved to all orders in the gauge couplings. As a consequence, nonperturbative effects from the gauge sector cannot induce $B - L$ violation. Since the neutrino mass operator described above violates also $B - L$, this proves that neutrino masses remain zero even in the presence of nonperturbative effects.

Let us now turn to the effect of gravity. Clearly, as long as we treat gravity in perturbation theory, the above symmetry arguments hold since all gravity coupling respect $B - L$ symmetry. However, once nonperturbative gravitational effects, e.g. black holes and worm holes, are included, there is no guarantee that global symmetries will be respected in the low-energy theory. The intuitive way to appreciate the argument is to note that throwing baryons into a black hole does not lead to any detectable consequence except through a net change in the baryon number of the universe. Since one can throw an arbitrary number of baryons into the black hole, an arbitrary information loss about the net number of missing baryons would prevent us from defining a baryon number of the visible universe—thus baryon number in the presence of a black hole cannot be an exact symmetry. Similar arguments can be made for any global

charge such as lepton number in the standard model. A field-theoretic parameterization of this statement is that the effective low-energy Lagrangian for the standard model in the presence of black holes and worm holes, etc must contain baryon and lepton number violating terms. In the context of the standard model, the only such terms that one can construct are nonrenormalizable terms of the form $LHLH/M_{Pl}$. After gauge-symmetry breaking, they lead to neutrino masses; however these masses are at most of order $v_{wk}^2/M_{Pl} \simeq 10^{-5}$ eV [36]. However, as we discussed in the previous section, to solve the atmospheric neutrino problem, one needs masses at least three orders of magnitude higher.

Thus, one must seek physics beyond the standard model to explain observed evidences for neutrino masses. Although there are many possibilities that lead to small neutrino masses of both Majorana as well as Dirac kind, here we focus on the possibility that there is a heavy right-handed neutrino (or neutrinos) that lead to a small neutrino mass. The resulting mechanism is known as the seesaw mechanism and leads to neutrino being a Majorana particle.

The nature and origin of the seesaw mechanism can also be tested in other experiments and we will discuss them below. This will be dependent on the kind of operators that play a role in generating neutrino masses. If the leading-order operator is of dimension 5, then the scale necessarily is very high (of order 10^{12} GeV or greater). On the other hand, in theories with extra space dimensions, this operator may be forbidden and one may be forced to go to higher-dimensional operators, in which case the scale could be lower. In the lecture IV, an example of five- and six-dimensional theory is given, where this indeed happens.

The seesaw mechanism raises a very important question: since we require the mass of the right-handed neutrino to be much less than the Planck scale, a key question is ‘what symmetry keeps the right-handed neutrino mass lighter?’ We will give two examples of symmetries that can do this.

2.2. Seesaw and the right-handed neutrino

The simplest possibility extension of the standard model that leads to nonzero mass for the neutrino is one where only a right-handed neutrino is added to the standard model. In this case ν_L and ν_R can form a mass term, but *a priori*, this mass term is like the mass terms for charged leptons or quark masses and will therefore involve the weak scale. If we call the corresponding Yukawa coupling to be Y_ν , then the neutrino mass is $m_D = Y_\nu v/\sqrt{2}$. For a neutrino mass in the eV range requires that $Y_\nu \simeq 10^{-11}$ or less. Introduction of such small coupling constants into a theory is generally considered unnatural and a sound theory must find a symmetry reason for such smallness. As already alluded to before, the seesaw mechanism [5], where we introduce a singlet Majorana mass term for the right-handed neutrino is one way to achieve this goal. What we have in this case is a (ν_L, ν_R) mass matrix which has the form

$$M = \begin{pmatrix} 0 & m_D \\ m_D & M_R \end{pmatrix}. \quad (43)$$

Since M_R is not constrained by the standard model symmetries, it is natural to choose it to be at a scale much higher than the weak scale. Now diagonalizing this mass matrix for a single neutrino species, we get a heavy eigenstate N_R with mass M_R and a light eigenstate ν with mass $m_\nu \simeq -m^2/M_R$. This provides a natural way to understand a small neutrino mass without any unnatural adjustment of parameters of a theory. In a subsequent section, we will discuss a theory which connects the scale M_R to a new symmetry of nature beyond the standard model.

2.2.1. *Why is $M_{\nu_R} \ll M_{P\ell}$?* The question ‘why $M_{\nu_R} \ll M_{P\ell}$?’ is in many ways similar to the question in the standard model is ‘why is $M_{\text{Higgs}} \ll M_{P\ell}$?’ It is well known that searches for answer to this question has led us to consider many interesting possibilities for physics beyond the standard model and supersymmetry appears to be the most promising answer to this question. It is hoped that answering this question for ν_R can also lead us to new insight into new symmetries beyond the standard model. There are two interesting answers to our question that I will elaborate later on.

$B - L$: If one adds three right-handed neutrinos to implement the seesaw mechanism, the model admits an anomaly free new symmetry, i.e. $B - L$. One can therefore extend the standard model symmetry to either $SU(2)_L \times U(1)_{I_{3R}} \times U(1)_{B-L}$ or its left–right symmetric extension $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. In either case, the right-handed neutrino carries the $B - L$ quantum number and its Majorana mass breaks this symmetry. Therefore, the mass of the ν_R can at most be the scale of $B - L$ symmetry breaking, hence answering the question ‘why $M_{\nu_R} \ll M_{P\ell}$?’.

$SU(2)_H$: While local $B - L$ is perhaps the most straightforward and natural symmetry that keeps ν_R lighter than the Planck scale, another possibility has recently been suggested in [24]. The main observation here is that if the standard model is extended by including a local $SU(2)_H$ symmetry acting on the first two lepton generations including the right-handed charged leptons, then global Witten anomaly freedom dictates that there must be at least two right-handed neutrinos which transform as a doublet under the $SU(2)_H$ local symmetry. In this class of models, in the limit of exact $SU(2)_H$ symmetry, the ν_{RS} are massless and as soon as the $SU(2)_H$ symmetry is broken, they pick up mass. Therefore, ‘lightness’ of the ν_{RS} compared with the Planck scale in these models is related to an $SU(2)_H$ symmetry. These comments are elaborated with explicit examples later on in this review.

2.2.2. *Small neutrino mass using a double seesaw mechanism with ν_R .* As we have seen from the previous discussion, the conventional seesaw mechanism requires rather high mass for the right-handed neutrino and therefore a correspondingly high scale for $B - L$ symmetry breaking. There is however no way at present to know what the scale of $B - L$ symmetry breaking is. There are, for example model bases on string compactification [37], where the $B - L$ is quite possibly in the TeV range. In this case small neutrino mass can be implemented by a double seesaw mechanism suggested in [38]. The idea is to take a right-handed neutrino N and a singlet neutrino S which has extra quantum numbers which prevent it from coupling with the left-handed neutrino. One can then write a 3×3 neutrino mass matrix in the basis (ν, N, S) of the form

$$M = \begin{pmatrix} 0 & m_D & 0 \\ m_D & 0 & M \\ 0 & M & \mu \end{pmatrix}. \quad (44)$$

For the case $\mu \ll M \approx M_{B-L}$ (where M_{B-L} is the $B - L$ breaking scale) this matrix has one light and two heavy neutrinos per generation and the latter two form a pseudo-Dirac pair with mass of the order of M_{B-L} . The important thing for us is that the light mass eigenvalue is given by $m_D^2 \mu / M^2$; for $m_D \approx \mu \simeq \text{GeV}$, a 10 TeV $B - L$ scale is enough to give neutrino masses in the eV range. For the case of three generations, the formula for the light neutrino mass matrix is given by

$$\mathcal{M}_\nu = M_D M^{-1} \mu M^{-1} M_D^T. \quad (45)$$

An example of a model where such a situation can arise is as follows: consider a supersymmetric model with the gauge group $SU(2)_L \times U(1)_{I_{3R}} \times U(1)_{B-L} \times U(1)_X$, with the leptonic sector assignment as $L(2, 0, -1)_0$, $\nu^c(1, -1/2, +1)_0$, $N(1, 0, 0)_{+1}$ and Higgs field assignments as $H_u(2, +1/2, 0)_0$, $\chi(1, +1/2, -1)_{-1}$, $\bar{\chi}(1, -1/2, +1)_1$, $\chi'(1, -1/2, +1)_{-1}$ and $\bar{\chi}'(1, +1/2, -1)_{+1}$. The relevant superpotential that is invariant under this is

$$W = h_\nu L H_u \nu^c + h_{\nu^c} \nu^c \chi N + f \frac{SS\chi\chi'}{M_H}. \quad (46)$$

If we give a vacuum expectation value (vev) to the $\chi\chi'$ and their conjugates of order 10 TeV and choose $M_H \sim 10^{12}$ GeV, then μ in the matrix is of the order of a 100 keV and for $m_D \sim 10$ GeV, we get the light neutrino spectrum of the right order. There are also terms in the superpotential such as $h' L H_u S \chi' / M_H$, which can induce mass terms of the order of a keV if $h' \sim 10^{-3}$ and it will not disturb the small neutrino mass prediction for these models. For further discussion of such effects in SO(10) models, see [39].

2.3. A gauge model for seesaw mechanism: left–right symmetric unification and type II seesaw

Let us now explore the implications of including the right-handed neutrinos into the extensions of the standard model to understand the small neutrino mass by the seesaw mechanism. As already emphasized, if we assume that there are no new symmetries beyond the standard model, the right-handed neutrino will have a natural mass of the order of the Planck scale making the light neutrino masses too small to be of interest in understanding the observed oscillations. We must therefore search for new symmetries that can keep the RH neutrinos at a lower scale than the Planck scale. A new symmetry always helps in making this natural.

To study this question, let us note that the inclusion of the right-handed neutrinos transforms the dynamics of the gauge models in a profound way. To clarify what we mean, note that in the standard model (that does not contain a ν_R) the $B - L$ symmetry is only linearly anomaly free, i.e. $\text{Tr}[(B - L)Q_a^2] = 0$, where Q_a are the gauge generators of the standard model but $\text{Tr}(B - L)^3 \neq 0$. This means that $B - L$ is only a global symmetry and cannot be gauged. However as soon as the ν_R is added to the standard model, one gets $\text{Tr}[(B - L)^3] = 0$ implying that the $B - L$ symmetry is now gaugeable and one could choose the gauge group of nature to be either $SU(2)_L \times U(1)_{I_{3R}} \times U(1)_{B-L}$ or $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, the latter being the gauge group of the left–right symmetric models [40]. Furthermore, the presence of the ν_R makes the model quark lepton symmetric and leads to a Gell–Mann–Nishijima-like formula for the electric charges [41], i.e.

$$Q = I_{3L} + I_{3R} + \frac{B - L}{2}. \quad (47)$$

The advantage of this formula over the charge formula in the standard model charge formula is that in this case all entries have a physical meaning. Furthermore, it leads naturally to Majorana nature of neutrinos as can be seen by looking at the distance scale where the $SU(2)_L \times U(1)_Y$ symmetry is valid but the left–right gauge group is broken. In that case, one gets

$$\Delta Q = 0 = \Delta I_{3L}, \quad \Delta I_{3R} = -\Delta \frac{B - L}{2}. \quad (48)$$

Table 3. Assignment of the fermion and Higgs fields to the representation of the left–right symmetry group.

Fields	$SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ representation
Q_L	$(2, 1, +\frac{1}{3})$
Q_R	$(1, 2, \frac{1}{3})$
L_L	$(2, 1, -1)$
L_R	$(1, 2, -1)$
ϕ	$(2, 2, 0)$
Δ_L	$(3, 1, +2)$
Δ_R	$(1, 3, +2)$

We see that if the Higgs fields that break the left–right gauge group carry right-handed isospin of 1, one must have $|\Delta_L| = 2$ which means that the neutrino mass must be Majorana type and the theory will break lepton number by two units.

Let us now proceed to give a few details of the left–right symmetric model and demonstrate how the seesaw mechanism emerges in this model.

The gauge group of the theory is $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ with quarks and leptons transforming as doublets under $SU(2)_{L,R}$. In table 3, we denote the quark, lepton and Higgs fields in the theory along with their transformation properties under the gauge group.

The first task is to specify how the left–right symmetry group breaks to the standard model, i.e. how one breaks the $SU(2)_R \times U(1)_{B-L}$ symmetry so that the success of the standard model including the observed predominant V-A structure of weak interactions at low energies is reproduced. Another question of naturalness that also arises simultaneously is that since the charged fermions and the neutrinos are treated completely symmetrically (quark–lepton symmetry) in this model, how does one understand the smallness of the neutrino masses compared with the other fermion masses.

It turns out that both the above problems of the LR model have a common solution. The process of spontaneous breaking of the $SU(2)_R$ symmetry that suppresses the V + A currents at low energies also solves the problem of ultralight neutrino masses. To see this let us write the Higgs fields explicitly:

$$\Delta = \begin{pmatrix} \Delta^+/\sqrt{2} & \Delta^{++} \\ \Delta^0 & -\Delta^+/\sqrt{2} \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1^0 & \phi_2^+ \\ \phi_1^- & \phi_2^0 \end{pmatrix}. \quad (49)$$

All these Higgs fields have Yukawa couplings to the fermions given symbolically as

$$\mathcal{L}_Y = h_1 \bar{L}_L \phi L_R + h_2 \bar{L}_L \tilde{\phi} L_R + h'_1 \bar{Q}_L \phi Q_R + h'_2 \bar{Q}_L \tilde{\phi} Q_R + f(L_L L_L \Delta_L + L_R L_R \Delta_R) + \text{h.c.} \quad (50)$$

The $SU(2)_R \times U(1)_{B-L}$ is broken down to the standard model hypercharge $U(1)_Y$ by choosing $\langle \Delta_R^0 \rangle = v_R \neq 0$ since this carries both $SU(2)_R$ and $U(1)_{B-L}$ quantum numbers. It gives mass to the charged and neutral right-handed gauge bosons, i.e. $M_{W_R} = gv_R$ and $M_{Z'} = \sqrt{2}gv_R \cos \theta_W / \sqrt{\cos 2\theta_W}$. Thus by adjusting the value of v_R one can suppress the right-handed current effects in both neutral and charged current interactions arbitrarily leading to an effective near maximal left-handed form for the charged current weak interactions.

The fact that at the same time the neutrino masses also become small can be seen by looking at the form of the Yukawa couplings. Note that the f-term leads to a mass for the right-handed neutrinos only at the scale v_R . Next, as we break the standard model symmetry by turning on the vev's for the ϕ fields as $\text{Diag}\langle\phi\rangle = (\kappa, \kappa')$, we not only give masses to the W_L and the Z bosons but also to the quarks and the leptons. In the neutrino sector, the above Yukawa couplings after $SU(2)_L$ breaking by $\langle\phi\rangle \neq 0$ lead to the so-called Dirac masses for the neutrino connecting the left- and right-handed neutrinos. In the two-component neutrino language, this leads to the following mass matrix for the ν, N (where we have denoted the left-handed neutrino by ν and the right-handed component by N):

$$M = \begin{pmatrix} 0 & h\kappa \\ h\kappa & fv_R \end{pmatrix}. \quad (51)$$

Note that m_D in previous discussions of the seesaw formula (see equation ()) is given by $m_D = h\kappa$, which links it to the weak scale and the mass of the RH neutrinos is given by $M_R = fv_R$, which is linked to the local $B - L$ symmetry. This justifies keeping RH neutrino mass at a scale lower than the Planck mass. It is therefore fair to assume that the seesaw mechanism coupled with observations of neutrino oscillations are a strong indication of the existence of a local $B - L$ symmetry far below the Planck scale.

By diagonalizing this 2×2 matrix, we get the light neutrino eigenvalue to be $m_\nu \simeq (h\kappa)^2/fv_R$ and the heavy one to be fv_R . Note that typical charged fermion masses are given by $h'\kappa$ etc. So since $v_R \gg \kappa, \kappa'$, the light neutrino mass is automatically suppressed. This way of suppressing the neutrino masses is called the seesaw mechanism [5]. Thus, in one stroke, one explains the smallness of the neutrino mass as well as the suppression of the V + A currents.

In deriving the above seesaw formula for neutrino masses, it has been assumed that the vev of the left-handed triplet is zero so that the $\nu_L \nu_L$ entry of the neutrino mass matrix is zero. However, in most explicit models such as the left-right model which provide an explicit derivation of this formula, there is an induced vev for the Δ_L^0 of order $\langle\Delta_L^0\rangle = v_T \simeq v_{wk}^2/v_R$. In the left-right models, this arises from the presence of a coupling in the Higgs potential of the form $\Delta_L \phi \Delta_R^\dagger \phi^\dagger$. In the presence of the Δ_L vev, the seesaw formula undergoes a fundamental change. One can have two types of seesaw formulae depending on whether the Δ_L has vev or not:

Type I seesaw formula:

$$M_\nu \simeq -M_D^T M_{N_R}^{-1} M_D, \quad (52)$$

where M_D is the Dirac neutrino mass matrix and $M_{N_R} \equiv fv_R$ is the right-handed neutrino mass matrix in terms of the Δ Yukawa coupling matrix f .

Type II seesaw formula [42]:

$$M_\nu \simeq f \frac{v_{wk}^2}{v_R} - M_D^T M_{N_R}^{-1} M_D. \quad (53)$$

Note that in the type I seesaw formula, what appears is the square of the Dirac neutrino mass matrix which is, in general, expected to have the same hierarchical structure as the corresponding charged fermion mass matrix. In fact, in some specific GUT models such as $SO(10)$, $M_D = M_u$. This is the origin of the common statement that neutrino masses given by the seesaw formula are hierarchical, i.e. $m_{\nu_e} \ll m_{\nu_\mu} \ll m_{\nu_\tau}$ and even a more model-dependent statement that $m_{\nu_e} : m_{\nu_\mu} : m_{\nu_\tau} = m_u^2 : m_c^2 : m_t^2$.

On the other hand, if one uses the type II seesaw formula, there is no reason to expect a hierarchy and in fact if the neutrino masses turn out to be degenerate as discussed before

as one possibility, one possible way to understand this may be to use the type II seesaw formula.

Secondly, the type II seesaw formula is a reflection of the parity invariance of the theory at high energies. Evidence for it would point more strongly towards left-right symmetry at high energies.

2.4. Understanding detailed pattern for neutrinos using the seesaw formula

Let us now address the question: to what extent can one understand the details of the neutrino masses and mixings using the seesaw formulae? The answer to this question is quite model-dependent. While there exist many models which fit the observations, none (except a few) are completely predictive and almost always they need to invoke new symmetries or new assumptions. The problem in general is that the seesaw formula of type I has 12 parameters in the absence of CP violation (six parameters for a symmetric Dirac mass matrix and six for the M_R) which is why its predictive power is so limited. In the presence of CP violation, the number of parameters double making the situation worse. Specific predictions can be made only under additional assumptions.

For instance, in a class of seesaw models based on the $SO(10)$ group that embodies the left-right symmetric unification model or the $SU(4)$ -colour, the mass the tau neutrino mass can be estimated provided one assumes the normal mass hierarchy for neutrinos and a certain parameter accompanying a higher-dimensional operator to be of order 1. To see this, let us assume that in the $SO(10)$ theory, the $B - L$ symmetry is broken by a **16**-dimensional Higgs boson. The RH neutrino mass in such a model arises from the nonrenormalizable operator $\lambda(\mathbf{16}_F \mathbf{16}_H)^2 / M_{Pl}$. In a supersymmetric theory, if **16**-Higgs is also responsible for GUT symmetry breaking, then after symmetry breaking, one obtains the RH neutrino mass $M_R \simeq \lambda(2 \times 10^{16})^2 / M_{Pl} \simeq 4\lambda \times 10^{14}$ GeV. In models with $SU(4)_c$ symmetry, $m_{\nu_{\tau,D}} \simeq m_t(M_U) \sim 100$ GeV. Using the seesaw formula then, one obtains for $\lambda = 1$, tau neutrino mass $m_{\nu_{\tau}} \simeq 0.025$ eV, which is close to the presently preferred value of 0.05 eV. The situation with respect to other neutrino masses is however less certain and here one has to make assumptions.

The situation with respect to mixing angles is much more complicated. For instance, the striking difference between the quark and neutrino mixing angles makes one doubt whether complete quark lepton unification is truly obeyed in nature. In sections 2.8 and 3.3, two examples are given where very few assumptions are made in getting maximal atmospheric mixing angle.

2.5. General consequences of the seesaw formula for neutrino masses

In this section, we will consider some implications of the seesaw mechanism for understanding neutrino masses. We will discuss two main points. One is the nature of the right-handed neutrino spectrum as dictated by the seesaw mechanism and, secondly, ways to get an approximate $L_e - L_\mu - L_\tau$ symmetric neutrino mass matrix using the seesaw mechanism and its possible implications for physics beyond the standard model.⁴

For this purpose, we use the type I seesaw formula along with the assumption of a diagonal Dirac neutrino mass matrix to obtain the right-handed neutrino mass matrix M_R :

$$\mathcal{M}_{R,ij} = m_{D,i} \mu_{ij}^{-1} m_{D,j} \quad (54)$$

⁴ For discussions of the seesaw formula and attempts to understand neutrino mixings see [54].

with

$$\begin{aligned}
\mu_{11}^{-1} &= \frac{c^2}{m_1} + \frac{s^2}{m_2} + \frac{\epsilon^2}{m_3}, \\
\mu_{12}^{-1} &= -\frac{c(s+c\epsilon)}{\sqrt{2}m_1} + \frac{s(c-s\epsilon)}{\sqrt{2}m_2} + \frac{\epsilon}{\sqrt{2}m_3}, \\
\mu_{13}^{-1} &= \frac{c(s-c\epsilon)}{\sqrt{2}m_1} - \frac{s(c+s\epsilon)}{\sqrt{2}m_2} + \frac{\epsilon}{\sqrt{2}m_3}, \\
\mu_{22}^{-1} &= \frac{(s+c\epsilon)^2}{2m_1} + \frac{(c_s\epsilon)^2}{2m_2} + \frac{1}{2m_3}, \\
\mu_{23}^{-1} &= -\frac{(s^2-c^2\epsilon^2)}{2m_1} - \frac{(c^2-s^2\epsilon^2)}{2m_2} + \frac{1}{2m_3}, \\
\mu_{33}^{-1} &= \frac{(s-c\epsilon)^2}{2m_1} + \frac{(c+s\epsilon)^2}{2m_2} + \frac{1}{2m_3}.
\end{aligned} \tag{55}$$

Since for the cases of normal and inverted hierarchy, we have no information on the mass of the lightest neutrino m_1 , we could assume it in principle to be quite small. In that case, the above equation enables us to conclude that quite probably one of the three right-handed neutrinos is much heavier than the other two. The situation is of course completely different for the degenerate case. This kind of separation of the RH neutrino spectrum is very suggestive of a symmetry. In fact, we have recently argued that [24], this indicates the possible existence of an $SU(2)_H$ horizontal symmetry, that leads in the simplest case to an inverted mass pattern for light neutrinos. This idea is discussed in a subsequent section.

2.6. $L_e-L_\mu-L_\tau$ symmetry and 3×2 seesaw

In this section, we discuss how an approximate $L_e-L_\mu-L_\tau$ symmetric neutrino mass matrix may arise within a seesaw framework. Consider a simple extension of the standard model by adding two additional singlet right-handed neutrinos [55], N_1, N_2 assigning them $L_e-L_\mu-L_\tau$ quantum numbers of +1 and -1 respectively. Denoting the standard model lepton doublets by $\psi_{e,\mu,\tau}$, the $L_e-L_\mu-L_\tau$ symmetry allows the following new couplings to the Lagrangian of the standard model:

$$\mathcal{L}' = (h_3\bar{\psi}_\tau + h_2\bar{\psi}_\mu)HN_2 + h_1\bar{\psi}_eHN_1 + MN_1^TC^{-1}N_2 + \text{h.c.}, \tag{56}$$

where H is the Higgs doublet of the standard model and C^{-1} is the Dirac charge conjugation matrix. We add to it the symmetry-breaking mass terms for the right-handed neutrinos, which are soft terms, i.e.

$$\mathcal{L}_B = \epsilon(M_1N_1^TC^{-1}N_1 + M_2N_2^TC^{-1}N_2) + \text{h.c.} \tag{57}$$

with $\epsilon \ll 1$. These terms break $L_e-L_\mu-L_\tau$ by two units but since they are dimension 3 terms, they are soft and do not induce any new terms into the theory.

It is clear from the resulting mass matrix for the ν_L, N system that the linear combination $(h_2\nu_\tau - h_3\nu_\mu)$ is massless and the atmospheric oscillation angle is given by $\tan \theta_A = h_2/h_3$; for

$h_3 \sim h_2$, the θ_A is maximal. The seesaw mass matrix then takes the following form (in the basis $(\nu_e, \tilde{\nu}_\mu, N_1, N_2)$ with $\tilde{\nu}_\mu \equiv h_2\nu_\mu + h_3\nu_\tau$):

$$M = \begin{pmatrix} 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_2 \\ m_1 & 0 & \epsilon M_1 & M \\ 0 & m_2 & M & \epsilon M_2 \end{pmatrix}. \quad (58)$$

The diagonalization of this mass matrix leads to the mass matrix of the form discussed before.

2.7. $SO(10)$ realization of the seesaw mechanism

The most natural grand unified theory for the seesaw mechanism is the $SO(10)$ model, although as has been mentioned, the gross dissimilarity between quark and lepton mixings makes additional assumptions necessary to reconcile with the inherent quark–lepton unification in such models. Nevertheless, the $SO(10)$ models are so natural framework for neutrino masses that some salient features may be instructive for any neutrino model building.

The first interesting point to note about the $SO(10)$ models is that the **16**-dimensional spinor representation contains all the fermions of each generation in the standard model plus the right-handed neutrino. Thus, the right-handed neutrino being necessary for the seesaw mechanism is automatic in these models. Secondly, in order to break the $B - L$ symmetry present in the $SO(10)$ group, one may use either the Higgs multiplets in **16**- or **126**-dimensional rep. We will see that either of these representations can be used to implement the seesaw mechanism. To see this note that under the left–right symmetric group $SU(2)_L \times SU(2)_R \times SU(4)_c$, these fields decompose as follows:

$$\begin{aligned} \mathbf{16} &= (2, 1, 4) \oplus (1, 2, 4^*), \\ \mathbf{126} &= (1, 1, 6) \oplus (3, 1, 10) \oplus (1, 3, 10^*) \oplus (2, 2, 15). \end{aligned} \quad (59)$$

Note that to break the $B - L$ symmetry it is the $(1, 2, 4^*)$ and $(1, 3, 10^*)$ in the respective multiplets whose neutral elements need to pick up a large vev. Note however that the $\mathbf{16}_H$ does not have any renormalizable coupling with the **16** spinors which contain the ν_R , whereas there is a renormalizable $SO(10)$ invariant coupling of the form $\mathbf{16}\mathbf{16}\mathbf{126}$ for the second multiplet. Therefore, if we decided to stay with the renormalizable model, then we would need a **126**-dimensional representation to implement the seesaw mechanism, whereas if we used the **16** to break the $B - L$ symmetry, we would require nonrenormalizable couplings of the form $\mathbf{16}^2\mathbf{16}^2/M_{pl}$. This has important implications for the $B - L$ scale. In the former case, the $B - L$ breaking scale is at an intermediate level such as $\sim 10^{13}$ GeV or so whereas in the latter case, we can have $B - L$ scale coincide with the GUT scale of 2×10^{16} GeV as in the typical SUSYGUT models.⁵

In addition to having the right-handed neutrino as part of the basic fermion representation and the Higgs representations the $SO(10)$ model has several other potential advantages for understanding neutrino masses. For example, if one uses only the **10**-dimensional representation

⁵ For a review of SUSY GUTs, see [49].

for giving masses to the quarks and leptons, one has the up-quark mass matrix M_u being equal to the Dirac mass matrix of the neutrinos which goes into the seesaw formula. As a result, if we work in a basis where the up-quark masses are diagonal so that all CKM mixings come from the down mass matrix, then the number of arbitrary parameters in the seesaw formula goes down from 12 to 6. Thus, even though one cannot predict neutrino masses and mixings, the parameters of the theory get fixed by their values as inputs. This may then be testable through its other predictions. The **10** only Higgs models have problems of their own, i.e. there are tree-level mass relations in the down sector such as $m_d/m_s = m_e/m_\mu$ which are renormalization group invariant and are in disagreement with observations. It may be possible in supersymmetric models to generate enough one-loop corrections out of the supersymmetry-breaking terms (nonuniversal) to save the situation but they will introduce unknown parameters into the theory.

*2.7.1. Breaking $B - L$ symmetry: **16** versus **126**.* To derive the standard model at low energies from $SO(10)$ symmetry, we must break the $B - L$ symmetry. The breaking of $B - L$ is responsible for the masses of the right-handed neutrino and is therefore intimately connected with neutrino masses via the seesaw mechanism. In an $SO(10)$ model, $B - L$ can be broken by either a **16**-dimensional Higgs boson or by a **126**-dimensional one. The former breaks $B - L$ by one unit whereas the latter does it by two. This has profound implications if the model is supersymmetric. To see this let us remind the reader that in the minimal supersymmetric standard model (MSSM), (unlike the standard model), the tree-level potential can break lepton and baryon number due to the presence of terms like QLd^c , LLe^c and $u^c d^c d^c$ in the superpotential. These terms go by the name of R-parity violating terms where R-parity symmetry is defined by $R_p = (-1)^{3(B-L)+2S}$ and S is the spin of the particle. These terms not only prevent the existence of stable dark matter, but they also lead to catastrophic decay rates for the proton.

In $SO(10)$ models where $B - L$ is broken by **16**-Higgs fields, there exists terms of the form $(\mathbf{16}_m)^3 \mathbf{16}_H$ where the subscript m, H denote matter and Higgs respectively. When $\mathbf{16}_H$ field breaks $B - L$, it does so by its ν_R -like component acquiring a vev. When this vev is substituted in the above dim four term in the superpotential, it generates the R-parity violating terms of MSSM. This can also be seen by looking at the formula for R-parity, where for $B - L = 1$, one gets $R_p = -1$ for the field that breaks $B - L$. Hence R-parity is broken.

On the other hand, if $B - L$ is broken by a **126** Higgs field, the only standard model singlet in this multiplet has $B - L = 2$. Therefore when it acquires a vev, it preserves R-parity and therefore no ‘unpleasant’ R-p violating terms at the MSSM level. One can also see this by writing all terms in the superpotential to all orders that involve **126** field and giving a vev to the $B - L = 2$ field in it to get the MSSM superpotential from it.

It is therefore clear that if one wants the lightest supersymmetric particle to be the dark matter candidate and one wants to work in an $SO(10)$ theory without specific extra symmetries, the better route to model building is to use the **126** Higgs to break $B - L$.

2.8. A minimal $SO(10)$ model for neutrinos with **126**

It was noted in [43] that if one considers a minimal $SO(10)$ model with a single **10** Higgs (denoted by H) and a single **126** Higgs (Δ) that couple with fermions, then the model not only leads to a naturally stable dark matter but also is remarkably predictive in the neutrino sector. This class of models has only two Yukawa coupling matrices: $h\psi\psi H + f\psi\psi\Delta$. Since $SO(10)$ symmetry implies that h and f are symmetric matrices, if we ignore CP violation, this model has nine

Yukawa coupling parameters. Since **10** and **126** have four Higgs doublets, in principle each one could have a nonzero vev. This leads to only 13 parameters that describe all quark and lepton masses. Noting further the fact that the Higgs standard model doublets in **126** transform as $(2, 2, 15)$ of $SU(2)_L \times SU(2)_R \times SU(4)_c$, one gets the quark and lepton mass matrices to have the following form:

$$\begin{aligned} M_\ell &= h\kappa_d - 3fv_d, \\ M_{\nu_D} &= h\kappa_u - 3fv_u, \end{aligned} \quad (60)$$

where $\kappa_{u,d}$ are the vev's of the up and down type Higgs doublets in the **10** Higgs and $v_{u,d}$ are the corresponding vevs for the standard model doublets in the **126** Higgs multiplet. Note that there are 13 parameters in the above equations and there are 13 inputs (six quark masses, three lepton masses and three quark mixing angles, mass of the Z-boson).

The next important observation on this model is that the same **126** responsible for the fermion masses also has a vev along the $\nu_R\nu_R$ directions so that it generates the right-handed neutrino mass matrix which are proportional to the f matrix. Thus, using the seesaw formula (type I), there are no free parameters in the light neutrino sector.

This model was extensively analysed in [43] before the emergence of all the neutrino oscillation data in late 1990s. It is now known that this minimal $SO(10)$ model without any CP phase cannot fit both the solar and the atmospheric neutrino data simultaneously and is therefore ruled out. It has however been recently noted that once the CP phases are properly included in the discussion, this model can yield a near-bimaximal mixing pattern for neutrinos [44]. However, this version has also now been ruled out.

It was noted in [45] for the case of two generations that if one assumes that the direct triplet term in type II seesaw dominates the seesaw formula, then it provides a very natural understanding of the large atmospheric mixing angle. The simple way to see it is to note that when the triplet term dominates the seesaw formula, then we have the neutrino mass matrix $\mathcal{M}_\nu \propto f$, where f matrix is the **126** coupling with fermions discussed earlier. Using the above equations, one can derive the following sum rule [46]:

$$\mathcal{M}_\nu = c(M_d - M_\ell). \quad (61)$$

Now quark lepton symmetry implies that for the second and third generations, the $M_{d,\ell}$ have the following general form:

$$M_d = \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_2 & m_b \end{pmatrix} \quad (62)$$

and

$$M_\ell = \begin{pmatrix} \epsilon'_1 & \epsilon'_2 \\ \epsilon'_2 & m_\tau \end{pmatrix}, \quad (63)$$

where $\epsilon_i \ll m_{b,\tau}$ as is required by low-energy observations. It is well known that in supersymmetric theories, when low-energy quark and lepton masses are extrapolated to the GUT scale, one gets approximately that $m_b \simeq m_\tau$. One then sees from the above sum rule for neutrino masses that all entries for the neutrino mass matrix are of the same order leading very naturally to the atmospheric mixing angle to be large.

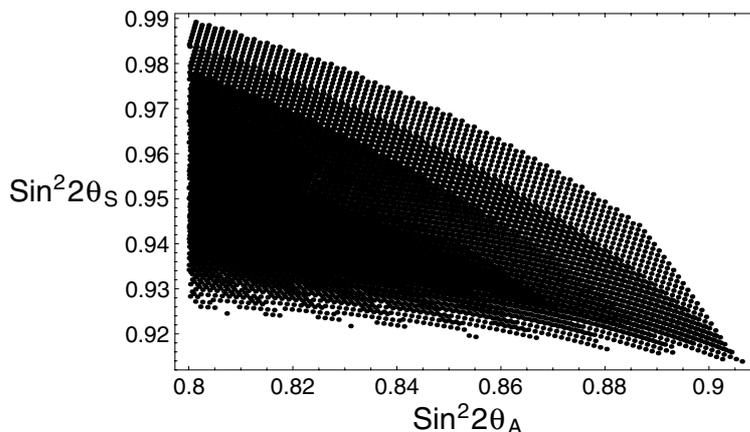


Figure 1. The range of predictions for $\sin^2 2\theta_\odot$ and $\sin^2 2\theta_A$ for the range of quark masses in table 2 that fit the charged lepton spectrum and where all CP phases are set to zero. It was required that the ratio $\Delta m_\odot^2/\Delta m_A^2 \leq 0.05$. Note that $\sin^2 2\theta_\odot \geq 0.9$ and $\sin^2 2\theta_A \leq 0.9$.

Since in this model there are no free parameters, it was not clear from the work of Bajc *et al* [45] that the solar mixing angle will fit observations. This question was addressed in [47], where it was shown that the same model can also lead to a large solar mixing angle. A simple way to see this is as follows.

Note that in the basis where the down-quark mass matrix is diagonal, all the quark mixing effects can be put in the down-quark mass matrix, i.e. $M_d = U_{CKM} M_d^d U_{CKM}^T$. Using the Wolfenstein parametrization for quark mixings, we can conclude that we have

$$M_d \approx m_b \begin{pmatrix} \lambda^4 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}. \quad (64)$$

In this model, there is a sum rule of the form $M_\ell = c_u M_u + c_d M_d$ which implies that M_ℓ and M_d have roughly similar pattern. In the above equation, the matrix elements are supposed to give only the approximate order of magnitude. As we extrapolate the quark masses to the GUT scale, due to the fact that $m_b - m_\tau \approx m_\tau \lambda^2$ for some value of $\tan \beta$, the neutrino mass matrix $M_\nu = c(M_d - M_\ell)$ takes roughly the form [21]

$$M_\nu = c(M_d - M_\ell) \approx m_0 \begin{pmatrix} \lambda^4 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & \lambda^2 \end{pmatrix}. \quad (65)$$

It is then easy to see from this mass matrix that both the θ_{12} (solar angle) and θ_{23} (the atmospheric angle) are large. It also turns out that the ratio of masses $m_2/m_3 \approx \lambda$, which explains the milder hierarchy among neutrinos compared with that among quarks. Mixing angle should be predicted and one has to see whether it is large. In fact, one such attempt before did give a small mixing angle, although there might be other domains of parameters where one may get a large solar mixing angle. Detailed calculations for this have shown that the model is quite consistent with present neutrino mixing data (see figures 1 and 2). Furthermore, it predicts the

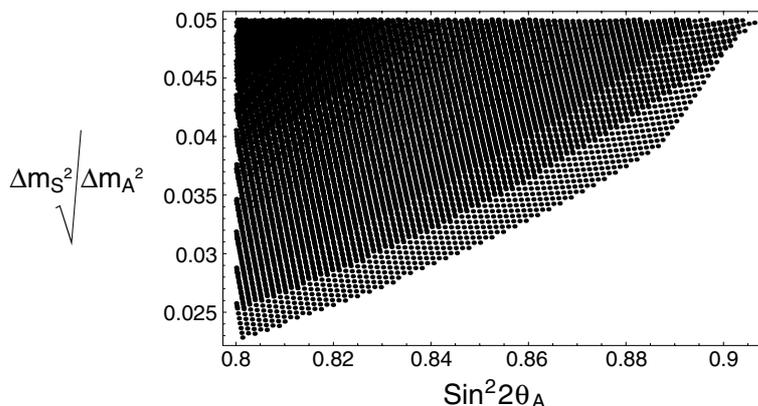


Figure 2. The predictions for $\sin^2 2\theta_A$ and $\Delta m_{\odot}^2 / \Delta m_A^2$ for the range of quark masses and mixings that fit charged lepton masses and where all CP phases in the fermion masses are set to zero.

$\theta_{13} = 0.16$ which can be tested in the MINOS as well as other planned long base line neutrino experiments.

Recently, other $SO(10)$ models have been considered where, under different assumptions, the atmospheric and solar neutrino data can be explained together [48].

2.9. Type II seesaw and quasi-degenerate neutrinos

In this subsection we will discuss some issues related to the degenerate neutrino hypothesis, which will be necessary if there is evidence for neutrinoless double beta decay at a significant level (see, for example, the recent results from the Heidelberg–Moscow group [13]) and assuming that no other physics such as R-parity breaking or doubly charged Higgs, etc are not the source of this effect). Thus, it is appropriate to discuss how such models can arise in theoretical schemes and how stable they are under radiative corrections.

There are two aspects to this question: one is whether the degeneracy arises within a gauge theory framework without arbitrary adjustment of parameters and the second aspect is that, given such a degeneracy arises at some scale naturally in a field theory, whether this mass degeneracy is stable under renormalization group extrapolation to the weak scale where we need the degeneracy to be present. In this section we comment on the former aspect.

It has already been alluded to before and first mentioned in [18] that degenerate neutrinos arise naturally in models that employ the type II seesaw since the first term in the mass formula is not connected to the charged fermion masses. One way that has been discussed is to consider schemes where one uses symmetries such as $SO(3)$ or $SU(2)$ or permutation symmetry S_4 [56] so that the Majorana Yukawa couplings f_i are all equal. This then leads to the dominant contribution to all neutrinos being equal. This symmetry however must be broken in the charged fermion sector to explain the observed quark and lepton masses. Such models consistent with known data have been constructed based on $SO(10)$ as well as other groups. The interesting point about the $SO(10)$ realization is that the dominant contributions to the Δm^2 s in this model comes from the second term in the type II seesaw formula which in simple models is hierarchical. It is of course known that if the MSW solution to the solar neutrino puzzle is the right solution (or an energy-independent solution), then we have $\Delta m_{solar}^2 \ll \Delta m_{ATMOS}^2$. In fact, if we use

the fact that $M_u = M_D$ holds in $SO(10)$ models then we have $\Delta m_{ATMOS}^2 \simeq m_0 m_i^2 / f v_R$ and $\Delta m_{SOLAR}^2 \simeq m_0 m_c^2 / f v_R$, where m_0 is the common mass for the three neutrinos. It is interesting that for $m_0 \sim \text{few eV}$ and $f v_R \approx 10^{15} \text{ GeV}$, both the Δm^2 s are close to the required values.

Outside the seesaw framework, there could also be electroweak symmetries that guarantee the mass degeneracy; see [20] for a recent model of this type.

The second question of stability under RGE of such a pattern is discussed in a subsequent section.

Section 3

3.1. Lepton flavour violation and neutrino masses

In the standard model, the masslessness of the neutrino implies that there is no lepton flavour changing effects unlike in the quark sector. Once one includes the right-handed neutrinos N_R , one for each family, there is lepton mixing and therefore lepton flavour changing effects such as $\mu \rightarrow e + \gamma$, $\tau \rightarrow e, \mu + \gamma$, etc. However, a simple estimate of the one-loop contribution to such effects shows that the amplitude is of the order

$$A(\ell_j \rightarrow \ell_i + \gamma) \simeq \frac{e G_F m_{\ell_j} m_e m_\nu^2}{\pi^2 m_W^2} \mu_B. \quad (66)$$

This leads to an unobservable branching ratio (of the order $\sim 10^{-40}$) for the rare radiative decay modes for the leptons.

The situation however changes drastically as soon as the seesaw mechanism for neutrino masses is embedded into the supersymmetric models. It has been noted in many papers already that in supersymmetric theories, the lepton flavour changing effects get significantly enhanced. They arise from the mixings among sleptons (superpartners of leptons) of different flavour caused by the renormalization group extrapolations which via loop diagrams lead to lepton flavour-violating (LFV) effects at low energies [58].

The way this happens is as follows. In the simplest $N = 1$ supergravity models [57], the supersymmetry-breaking terms at the Planck scale are taken to have only few parameters: a universal scalar mass m_0 , universal A terms and one gaugino mass $m_{1/2}$ for all three types of gauginos. Clearly, a universal scalar mass implies that at the Planck scale, there is no flavour violation anywhere except in the Yukawa couplings (or when the Yukawa terms are diagonalized in the CKM angles). However, as we extrapolate this theory to the weak scale, the flavour mixings in the Yukawa interactions induce nonuniversal flavour-violating scalar mass terms (i.e. flavour-violating slepton and squark mass terms). In the absence of neutrino masses, the Yukawa matrices for leptons can be diagonalized so that there is no flavour violation in the lepton sector even after extrapolation down to the weak scale. On the other hand, when neutrino mixings are present, there is no basis where all leptonic flavour mixings can be made to disappear. In fact, in the most general case, of the three matrices Y_ℓ , the charged lepton coupling matrix, Y_ν , RH neutrino Yukawa coupling and M_{N_R} , the matrix characterizing the heavy RH neutrino mixing, only one can be diagonalized by an appropriate choice of basis and the flavour mixing in the other two remain. In a somewhat restricted case where the right-handed neutrinos do not have any interaction other than the Yukawa interaction and an interaction that generates the Majorana mass for the right-handed neutrino, one can only diagonalize two out of the three matrices (i.e. Y_ν , Y_ℓ and M_R). Thus, there will always be lepton flavour-violating terms in the basic Lagrangian, no matter what

basis one chooses. These LFV terms can then induce mixings between the sleptons of different flavour and lead to LFV processes. If we keep the M_ℓ diagonal by choice of basis, searches for LFV processes such as $\tau \rightarrow \mu + \gamma$ and/or $\mu \rightarrow e + \gamma$ can throw light on the RH neutrino mixings/or family mixings in M_D , as has already been observed.

Since in the absence of CP violation, there are at least six mixing angles (nine if M_D is not symmetric) in the seesaw formula and only three are observable in neutrino oscillation, to get useful information on the fundamental high scale theory from LFV processes, it is assumed that M_{NR} is diagonal so that one has a direct correlation between the observed neutrino mixings and the fundamental high scale parameters of the theory. The important point is that the flavour mixings in Y_ν then reflect themselves in the slepton mixings that lead to the LFV processes via the RGEs.

From the point of view of the LFV analysis, there are essentially two classes of neutrino mass models that need to be considered: (i) the first class is where it is assumed that the RH neutrino mass M_{NR} is either a mass term in the basic Lagrangian or arises from nonrenormalizable terms such as $\nu^c \chi^{c2}/M_{Pl}$, as in a class of $SO(10)$ models; and (ii) a second class where the Majorana mass of the right-handed neutrino itself arises from a renormalizable Yukawa coupling, e.g. $f\nu^c\nu^c\Delta$. In the first class of models, in principle, one could decide to have all the flavour mixing effects in the right-handed neutrino mass matrix and keep the Y_ν diagonal. In that case, RGEs would not induce any LFV effects. However, we will bar this possibility and consider the case where all flavour mixings are in the Y_ν so that RGEs can induce LFV effects and estimate them in what follows. In class two models on the other hand, there will always be an LFV effect, although its magnitude will depend on the choice of the seesaw scale (v_{BL}).

Examples of class two models are models for neutrino mixings such as $SO(10)$ with a **126** Higgs field [43] or left–right model with a triplet Higgs, whose vev is the seesaw scale.

In both these examples, the key equations that determine the extent of lepton flavour violation are:

Case (i):

$$\begin{aligned} \frac{dm_L^2}{dt} &= \frac{1}{4\pi^2} [(m_L^2 + 2m_{H_d}^2)Y_\ell^\dagger Y_\ell + (m_L^2 + 2H_u^2)Y_\nu^\dagger Y_\nu + 2Y_\ell^\dagger m_{ec}^2 Y_\ell + Y_\ell^\dagger Y_\ell m_\ell^2 \\ &\quad + 2Y_\nu^\dagger m_{\nu^c}^2 Y_\nu + Y_\nu^\dagger Y_\nu m_\ell^2 + 2A_\ell^\dagger A_\ell + 2A_\nu^\dagger A_\nu - f(g^2)], \\ \frac{dA_\ell}{dt} &= \frac{1}{16\pi^2} A_\ell [\text{Tr}(3Y_d^\dagger Y_d + Y_\ell^\dagger Y_\ell) + 5Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu - 3g_2^2 - ag_R^2 - bg_{B-L}^2] \\ &\quad + Y_\ell [\text{Tr}(6A_d Y_d^\dagger + A_\ell Y_\ell^\dagger) + 4Y_\ell^\dagger A_\ell + 2Y_\nu^\dagger A_\nu + 6g_2^2 M_2 + \dots]. \end{aligned} \quad (67)$$

Case (ii): In addition to the above two equations, two more equations are necessary:

$$\begin{aligned} \frac{dY_\nu}{dt} &= \frac{Y_\nu}{16\pi^2} [\text{Tr}(3Y_u Y_u^\dagger + Y_\nu Y_\nu^\dagger) + 3Y_\nu^\dagger Y_\nu + Y_\ell^\dagger Y_\ell + 4f^\dagger f - 3g_2^2 - c_R g_R^2 - c_{B-L} g_{B-L}^2], \\ \frac{dA_\nu}{dt} &= \frac{1}{16\pi^2} A_\nu [\text{Tr}(3Y_u^\dagger Y_u + Y_\nu^\dagger Y_\nu) + 5Y_\nu^\dagger Y_\nu + Y_\ell^\dagger Y_\ell + 4f^\dagger f - 3g_2^2 - ag_R^2 - bg_{B-L}^2] \\ &\quad + Y_\ell [\text{Tr}(6A_d Y_d^\dagger + A_\ell Y_\ell^\dagger) + 4Y_\ell^\dagger A_\ell + 6g_2^2 M_2 + \dots]. \end{aligned} \quad (68)$$

To apply these equations, we note that in the basis where the charged lepton masses are diagonal, the seesaw formula involves the right-handed neutrino mass matrix M_R and the neutrino Dirac mass matrix M_D . Assuming the Dirac mass matrix is symmetric, there are 12 parameters (for the case with CP conservation). Since neutrino masses and mixings only provide six observables, there are several different ways that can lead to the observed neutrino mixings. Two distinct extreme ways are as follows: (i) the first case is where the neutrino mixings arise primarily from the off-diagonal elements of the M_D assuming the M_R is diagonal or even an extremely simplified case where it is a unit matrix and (ii) a second case where we can keep the M_D diagonal and all mixings arise from M_R having mixings. In the first case, the RGEs always lead to lepton flavour violation, whereas in the second case, flavour violations arise only if the M_R arises from a Majorana Yukawa coupling of the form $\nu^c \nu^c \Delta$ after $\langle \Delta \rangle \neq 0$ as already explained. We will call this the Majorana case and case (i) as the Dirac case.

In the Dirac case, starting with the simplest supersymmetry breaking assumption of universal scalar masses and proportional A terms, the scalar sleptons develop off-diagonal terms due to the flavour violation in Y_ν and these mass terms have the form

$$m_{L,ij}^2 \propto \frac{3+a^2}{16\pi^2} \ln \frac{M_{P\ell}}{M_{B-L}} Y_\nu^\dagger Y_\nu + \dots, \quad (69)$$

where \dots denote the diagonal terms that cannot cause flavour mixing. If the M_R is diagonal, the $Y_\nu^\dagger Y_\nu$ is nothing but the neutrino mass matrix upto a constant $= M_R$. Using this we can get the Dirac mass dependence of the $B(\ell_j \rightarrow e + \gamma)$ (for $\ell_j = \tau, \mu$) to be

$$B(\ell_j \rightarrow e + \gamma) \sim \frac{(m_2 c_\tau + m_3 \epsilon)^2 v_{B-L}^2}{G_F^2 v_{wk}^4 m_0^4} c_{\ell_j}, \quad (70)$$

where $c_\tau = 1/6$ and $c_\mu = 1$, whereas for $B(\tau \rightarrow \mu + \gamma) \propto 6m_3^2 v_{B-L}^2 / 4G_F^2 v_{wk}^4 m_0^4$.

Now coming to the second case with Majorana–Yukawa couplings, starting with universal scalar masses at the Planck scale and in the Majorana–Yukawa case all couplings flavour diagonal except the f coupling, it is easy to see that the above equations will induce flavour changing effects in m_L^2 and A_ℓ . The strength of the slepton flavour mixings depends sensitively on the neutrino mass texture and the resulting texture in the coupling matrix f . Below we give the branching ratio expressions for one extreme case where we assume that the lightest neutrino mass dominates the f matrix elements. We caution the reader that this is by no means the most typical case and results differ significantly as different mass textures are considered [26].

The slepton flavour mixings, via a one-loop diagram involving the winos (\tilde{W}^+ and \tilde{W}^3) lead to the lepton flavour changing radiative amplitudes. Keeping only the contribution of the m_L^2 term which dominates for larger $\tan \beta$, we find that roughly speaking the three branching ratios are given by

$$B(\mu \rightarrow e + \gamma) \propto \frac{1}{G_F^2 m_0^4 v_{B-L}^4} \frac{m_{D3}^4 m_{D2}^6 m_{D1}^2 \tan^2 \beta}{m_1^4 v_{wk}^4}, \quad (71)$$

$$B(\tau \rightarrow e + \gamma) \propto \frac{1}{G_F^2 m_0^4 v_{B-L}^4} \frac{m_{D3}^{10} m_{D1}^2 \tan^2 \beta}{m_1^4 v_{wk}^4}, \quad (72)$$

$$B(\tau \rightarrow \mu + \gamma) \propto \frac{1}{G_F^2 m_0^4 v_{B-L}^4} \frac{m_{D3}^{10} m_{D2}^2 \tan^2 \beta}{m_1^4 v_{wk}^4}. \quad (73)$$

Note an important difference between the predictions for the Dirac case and the Majorana one. Both $\tau \rightarrow e + \gamma$ and $\mu \rightarrow e + \gamma$ are of the same order of magnitude for the Dirac case, whereas they are very different for the Majorana case where $B(\mu \rightarrow e + \gamma)/B(\tau \rightarrow e + \gamma) \simeq (m_{D2}/m_{D3})^6$. Therefore, even a mild hierarchy in the Dirac mass sector can make the $\tau \rightarrow e + \gamma$ branching ratio much larger than the $\mu \rightarrow e + \gamma$ branching ratio. This kind of discrepancy could, in principle, be used to test the origin of the seesaw mechanism.

It must be pointed out that the predictions for the Majorana case are extremely texture-sensitive. In a recent paper [26], calculations have been carried out for a different but consistent texture where $B(\mu \rightarrow e + \gamma)$, although much lower than the τ rare decay branching ratios is still found to be within the range of currently planned experiments at PSI.

For completeness, we give the formula for calculating the radiative decay of the leptons. If we express the amplitude for the decay as

$$\mathcal{L} = iem_j(\bar{\ell}_{jL}\sigma_{\mu\nu}\ell_{iR}C_L + \bar{\ell}_{jR}\sigma_{\mu\nu}\ell_{iL}C_R)F^{\mu\nu} + \text{h.c.}, \quad (74)$$

then the branching ratio for the decay $\ell_j \rightarrow \ell_i + \gamma$ is given by the formula

$$B(\ell_j \rightarrow \ell_i + \gamma) = \frac{48\pi^3\alpha_{em}}{G_F^2}(|C_L|^2 + |C_R|^2)B(\ell_j \rightarrow \ell_i + 2\nu). \quad (75)$$

3.2. Renormalization group evolution of the neutrino mass matrix

In the seesaw models for neutrino masses, the neutrino mass arises from the effective operator

$$\mathcal{O}_\nu = -\frac{1}{4}\kappa_{\alpha\beta}\frac{L_\alpha H L_\beta H}{M} \quad (76)$$

after symmetry breaking $\langle H^0 \rangle \neq 0$; here L and H are the leptonic and weak doublets, respectively. α and β denote the weak flavour index. The matrix κ becomes the neutrino mass matrix after symmetry breaking, i.e. $\langle H^0 \rangle \neq 0$. This operator is defined at the scale M since it arises after the heavy field N_R is integrated out. On the other hand, in conventional oscillation experiments, the neutrino masses and mixings being probed are at the weak scale. One must therefore extrapolate the operator down from the seesaw scale M to the weak scale M_Z [59]. The form of the renormalization group extrapolation of course depends on the details of the theory. For simplicity, we will consider only the supersymmetric theories, where the only contributions come from the wave function renormalization and is therefore easy to calculate. The equation governing the extrapolation of the $\kappa_{\alpha\beta}$ matrix is given in the case of MSSM by

$$\frac{d\kappa}{dt} = [-3g_2^2 + 6\text{Tr}(Y_u^\dagger Y_u)]\kappa + \frac{1}{2}[\kappa(Y_e^\dagger Y_e) + (Y_e^\dagger Y_e)\kappa]. \quad (77)$$

We note two kinds of effects on the neutrino mass matrix from the above formula: (i) one that is flavour-independent and (ii) a part that is flavour-specific. If we work in a basis where the charged leptons are diagonal, then the resulting correction to the neutrino mass matrix is given by

$$\mathcal{M}_\nu(M_Z) = (1 + \delta)\mathcal{M}(M_{B-L})(1 + \delta), \quad (78)$$

where δ is a diagonal matrix with matrix elements $\delta_{\alpha\alpha} \simeq -m_\alpha^2 \tan^2 \beta / 16\pi^2 v^2$. In more complicated theories, the corrections will be different. Let us now study some implications of these corrections. For this first note that in the MSSM, this effect can be sizeable if $\tan \beta$ is large (of the order 10 or greater).

3.3. Radiative magnification of neutrino mixing angles

A major puzzle of quark lepton physics is the diverse nature of the mixing angles. Whereas in the quark sector the mixing angles are small, for the neutrinos they are large. One possible suggestion in this connection is that perhaps the mixing angles in both quark and lepton sectors at similar at some high scale; however, due to renormalization effects, they may become magnified at low scales. It was shown in [60] that this indeed happens if the neutrino spectrum is degenerate. This can be seen in a simple way for the $\nu_\mu - \nu_\tau$ sector [60].

Let us start with the mass matrix in the flavour basis:

$$\mathcal{M}_{\mathcal{F}} = \mathcal{U}^* \mathcal{M}_{\mathcal{D}} \mathcal{U}^\dagger = \begin{pmatrix} C_\theta & S_\theta \\ -S_\theta & C_\theta \end{pmatrix} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 e^{-i\phi} \end{pmatrix} \begin{pmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{pmatrix}. \quad (79)$$

Let us examine the situation when $\phi = 0$ (i.e. CP is conserved), which corresponds to the case when the neutrinos ν_1 and ν_2 are in the same CP eigenstate. Due to the presence of radiative corrections to m_1 and m_2 , the matrix $\mathcal{M}_{\mathcal{F}}$ gets modified to

$$\mathcal{M}_{\mathcal{F}} \rightarrow \begin{pmatrix} 1 + \delta_\alpha & 0 \\ 0 & 1 + \delta_\beta \end{pmatrix} \mathcal{M}_{\mathcal{F}} \begin{pmatrix} 1 + \delta_\alpha & 0 \\ 0 & 1 + \delta_\beta \end{pmatrix}. \quad (80)$$

The mixing angle $\bar{\theta}$ that now diagonalizes the matrix $\mathcal{M}_{\mathcal{F}}$ at the low scale μ (after radiative corrections) can be related to the old mixing angle θ by the following expression:

$$\tan 2\bar{\theta} = \tan 2\theta (1 + \delta_\alpha + \delta_\beta) \frac{1}{\lambda}, \quad (81)$$

where

$$\lambda \equiv \frac{(m_2 - m_1)C_{2\theta} + 2\delta_\beta(m_1 S_\theta^2 + m_2 C_\theta^2) - 2\delta_\alpha(m_1 C_\theta^2 + m_2 S_\theta^2)}{(m_2 - m_1)C_{2\theta}}. \quad (82)$$

If

$$(m_1 - m_2)C_{2\theta} = 2\delta_\beta(m_1 S_\theta^2 + m_2 C_\theta^2) - 2\delta_\alpha(m_1 C_\theta^2 + m_2 S_\theta^2), \quad (83)$$

then $\lambda = 0$ or equivalently $\bar{\theta} = \pi/4$, i.e. maximal mixing. Given the mass heirarchy of the charged leptons, $m_{l_\alpha} \ll m_{l_\beta}$, we expect $|\delta_\alpha| \ll |\delta_\beta|$, which reduces (83) to a simpler form:

$$\epsilon \equiv \frac{\delta m C_{2\theta}}{(m_1 S_\theta^2 + m_2 C_\theta^2)}. \quad (84)$$

For MSSM, the radiative magnification condition can be satisfied provided

$$h_\tau(\text{MSSM}) \approx \sqrt{\frac{8\pi^2 |\Delta m^2(\Lambda)| C_{2\theta}}{\ln(\Lambda/\mu) m^2}}. \quad (85)$$

For $\Delta m^2 \simeq \Delta m_A^2$, this condition can be satisfied for a very wide range of $\tan \beta$.

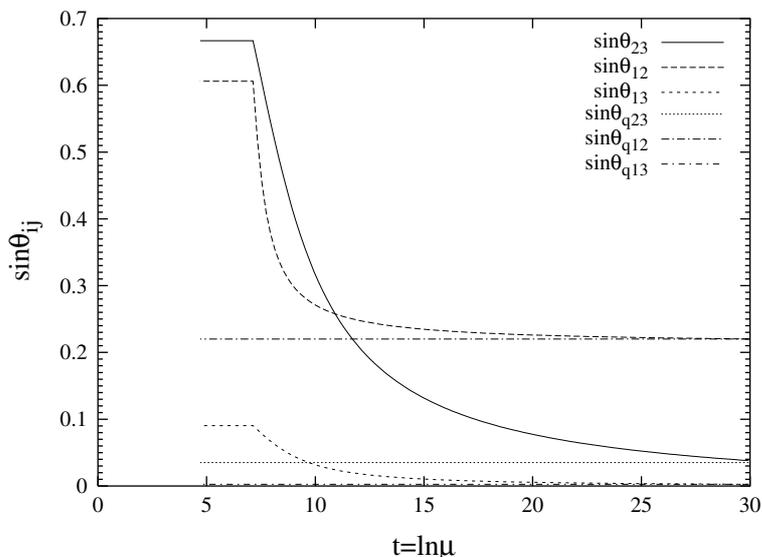


Figure 3. Evolution of small quark-like mixings at the seesaw scale to bi-large neutrino mixings at low energies for the seesaw scale $M_R = 10^{13}$ GeV with $\tan \beta = 55$, $M_{\text{SUSY}} = 1$ TeV and mass eigenvalues and mixing angles as given in the first column of table 1. The solid, long-dashed and short-dashed lines represent $\sin \theta_{23}$, $\sin \theta_{12}$ and $\sin \theta_{13}$, respectively, as defined in the text. The evolution of sines of quark mixing angles, $\sin \theta_{qij}$ ($i, j = 1, 2, 3$), are presented by almost horizontal lines.

A particularly intriguing result was recently found in [61], where it was shown that if one sets the neutrino and quark masses to be equal at the seesaw scale and if the quark masses are quasi-degenerate, then as the mixing angles are extrapolated to the weak scale, they get ‘magnified’ in such a way that the solar and the atmospheric angles become large and the θ_{13} remains within the present experimental limit. The detailed predictions are given in figure 3.

An important criterion for this to happen is that the common mass for all neutrinos must be above 0.1 eV, a value which can be detected in the planned double beta decay experiments.

It is important to emphasize that this magnification occurs only if at the seesaw scale the neutrino masses are nearly degenerate. A similar mechanism using the right-handed neutrino Yukawa couplings instead of the charged leptons has been carried out recently [62]. Here two conditions must be satisfied: (i) the neutrino spectrum must be nearly degenerate (i.e. $m_1 \simeq m_2$ as in [60]) and (ii) there must be a hierarchy between the right-handed neutrinos.

3.4. Generation of solar mass difference square in the case of $L_e-L_\mu-L_\tau$ symmetric models

In this section, we give another example where the renormalization group evolution (RGE) of the neutrino mass matrix plays an important role. Consider the following neutrino mass matrix defined at a (unification) scale much above the weak scale as in equation (6):

$$\mathcal{M}_\nu^0 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} m \quad (86)$$

Note that as already emphasized earlier, this matrix has several deficits as far as accommodating neutrino data is concerned. To remind the reader, the problems are twofold: (i) it leads to $\Delta m_{\odot}^2 = 0$ and (ii) solar neutrino mixing angle is maximal. As we show below, RGEs can help to overcome both these defects.

The important point to note is that the RGE depends on the charged lepton mass matrix. Let us assume that the charged lepton mass matrix has the form

$$\mathcal{M}_{\ell^+} = \begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ x' & 0 & 1 \end{pmatrix} m_{\tau}. \quad (87)$$

For the most part we will take $x' = x$ so that $|x| \simeq \sqrt{m_e/m_{\tau}}$ and $y \simeq m_{\mu}/m_{\tau}$. We will also consider the possibility that $x \gg x'$ (the right-handed singlet leptons multiply the matrix in this equation on the right). In the case $x = x'$; note that there is only a parameter in the leptonic sector (both the charged leptons and neutrinos). We will show that this model can lead to a realistic description of the neutrino oscillations.

At scales below the unification scale, through the renormalization of the effective $d = 5$ neutrino mass operator the form of M_{ν}^0 [30] will be modified. (Analogous corrections in M_{ℓ^+} is negligible.) The modified neutrino mass matrix at the weak scale is given by

$$\mathcal{M}_{\nu} \simeq \mathcal{M}_{\nu}^0 + \frac{c}{16\pi^2} \ln(M_U/M_Z) (Y_{\ell} Y_{\ell}^{\dagger} M_{\nu}^0 + M_{\nu}^0 (Y_{\ell} Y_{\ell}^{\dagger})^{\text{T}}). \quad (88)$$

Here $c = -3/2$ for SM while $c = 1$ for SUSY, and Y_{ℓ} is the charged lepton Yukawa coupling matrix. We have absorbed the flavour-independent renormalization factor into the definition of m in the above equation. Explicitly,

$$\mathcal{M}_{\nu} \simeq \begin{pmatrix} 2rx & 1 + r(x^2 + y^2) & 1 + r(1 + 2x^2) \\ 1 + r(x^2 + y^2) & 0 & rx \\ 1 + r(1 + 2x^2) & rx & 2rx \end{pmatrix} m. \quad (89)$$

This mass matrix has both an acceptable Δm_{\odot}^2 as well as large (but not maximal) solar neutrino mixing angle. This is another example where the renormalization group equations do make a difference in our understanding of the neutrino oscillations.

3.5. A horizontal symmetry approach to near bimaximal mixing

In this section, I present a model [24] which motivates the existence of an $SU(2)_H$ horizontal symmetry acting on leptons to understand the near bimaximal mixing pattern and yields the softly broken $L_e-L_{\mu}-L_{\tau}$ model discussed earlier.

Suppose there is an $SU(2)_H$ horizontal symmetry that acts only on leptons. As already discussed, freedom from global Witten anomaly requires that there must be two right-handed neutrinos that transform as a doublet of $SU(2)_H$. The local $SU(2)_H$ symmetry then implies that the masses of these two right-handed neutrinos are protected and must be at the scale of $SU(2)_H$ breaking. If there is a third right-handed neutrino for reasons of quark lepton symmetry, then it will acquire a mass of the order of the Planck or string scale and decouple

from neutrino physics at lower energies. This therefore provides a physically distinct way of implementing the seesaw mechanism. One has a 3×2 seesaw rather than the usual 3×3 one.

Furthermore, the $SU(2)$ horizontal symmetry restricts both the Dirac mass of the neutrino as well as the right-handed neutrino mass matrix to the forms [24]

$$M_{\nu_L, \nu_R} = \begin{pmatrix} 0 & 0 & 0 & h_0 \kappa_0 & 0 \\ 0 & 0 & 0 & 0 & h_0 \kappa_0 \\ 0 & 0 & 0 & h_1 \kappa_1 & h_1 \kappa_2 \\ h_0 \kappa_0 & 0 & h_1 \kappa_1 & 0 & f v'_H \\ 0 & h_0 \kappa_0 & h_1 \kappa_2 & f v'_H & 0 \end{pmatrix}. \quad (90)$$

After seesaw diagonalization, it leads to the light neutrino mass matrix of the form

$$\mathcal{M}_\nu = -M_D M_R^{-1} M_D^T, \quad (91)$$

where

$$M_D = \begin{pmatrix} h_0 \kappa_0 & 0 \\ 0 & h_0 \kappa_0 \\ h_1 \kappa_1 & h_1 \kappa_2 \end{pmatrix}, \quad M_R^{-1} = \frac{1}{f v'_H} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The resulting light Majorana neutrino mass matrix \mathcal{M}_ν is given by

$$\mathcal{M}_\nu = -\frac{1}{f v'_H} \begin{pmatrix} 0 & (h_0 \kappa_0)^2 & h_0 h_1 \kappa_0 \kappa_2 \\ (h_0 \kappa_0)^2 & 0 & h_0 h_1 \kappa_0 \kappa_1 \\ h_0 h_1 \kappa_0 \kappa_2 & h_0 h_1 \kappa_0 \kappa_1 & 2 h_1^2 \kappa_1 \kappa_2 \end{pmatrix}. \quad (92)$$

To get the physical neutrino mixings, we also need the charged lepton mass matrix defined by $\bar{\psi}_L \mathcal{M}_\ell \psi_R$. This is given in our model by

$$\mathcal{M}_\ell = \begin{pmatrix} h'_2 \kappa_0 & 0 & -h'_1 \kappa_2 \\ 0 & h'_2 \kappa_0 & h'_1 \kappa_1 \\ h'_4 \kappa_1 & h'_4 \kappa_2 & h'_3 \kappa_0 \end{pmatrix}. \quad (93)$$

Note that in the limit of $\kappa_1 = 0$, the neutrino mass matrix has exact $(L_e - L_\mu - L_\tau)$ symmetry, whereas the charged lepton mass matrix breaks this symmetry. This is precisely the class of inverted hierarchy models that was discussed earlier which provides a realistic as well as a testable model for neutrino oscillations. In particular, this model leads to a relation between the neutrino parameters U_{e3} and the ratio of solar and atmospheric mass difference squared, i.e.

$$U_{e3}^2 \cos 2\theta_\odot = \frac{\Delta m_\odot^2}{2\Delta m_A^2} + \mathcal{O}(U_{e3}^4, m_e/m_\mu), \quad (94)$$

which is testable in proposed long baseline experiments such as NUMI off-axis plan at Fermilab or JHF in Japan.

Section 4

4.1. Neutrino masses in models with large extra dimensions

One of the important predictions of string theories is the existence of more than three space dimensions. For a long time, it was believed that these extra dimensions are small and are therefore practically inconsequential as far as low-energy physics is concerned. However, recent progress in the understanding of the nonperturbative aspects of string theories has opened up the possibility that some of these extra dimensions could be large [66, 67] without contradicting observations. In particular, models where some of the extra dimensions have sizes as large as a millimetre and where the string scale is in the few TeV range have attracted a great deal of phenomenological attention in the past two years [66]. The basic assumption of these models, inspired by the D-branes in string theories, is that the space–time has a brane–bulk structure, where the brane is the familiar $(3 + 1)$ -dimensional space–time, with the standard model particles and forces residing in it, and the bulk consists of all space dimensions where gravity and other possible gauge singlet particles live. One could of course envision $(3 + d + 1)$ -dimensional D-branes where d -space dimensions have miniscule ($\leq \text{TeV}^{-1}$) size. The main interest in these models has been due to the fact that the low string scale provides an opportunity to test them using existing collider facilities.

A major challenge to these theories comes from the neutrino sector, the first problem being how one understands the small neutrino masses in a natural manner. The conventional seesaw [5] explanation which is believed to provide the most satisfactory way to understand this requires that the new physics scale (or the scale of $SU(2)_R \times U(1)_{B-L}$ symmetry) be around 10^9 – 10^{12} GeV or higher, depending on the Dirac masses of the neutrinos whose magnitudes are not known. If the highest scale of the theory is a TeV, clearly the seesaw mechanism does not work, so one must look for alternatives. The second problem is that if one considers only the standard model group in the brane, operators such as $LHLH/M_*$ could be induced by string theory in the low-energy-effective Lagrangian. For TeV scale strings this would obviously lead to unacceptable neutrino masses.

One mechanism suggested in [68] is to postulate the existence of one or more gauge singlet neutrinos, ν_B , in the bulk which couple with the lepton doublets in the brane. After electroweak symmetry breaking, this coupling can lead to neutrino Dirac masses, which are suppressed by the ratio M_*/M_{Pl} , where M_{Pl} is the Planck mass and M_* is the string scale. This is sufficient to explain small neutrino masses and owes its origin to the large bulk volume that suppresses the effective Yukawa couplings of the Kaluza–Klein (KK) modes of the bulk neutrino to the brane fields. In this class of models, naturalness of small neutrino mass requires that one must assume the existence of a global $B - L$ symmetry in the theory, since that will exclude the undesirable higher-dimensional operators from the theory.

To discuss the mechanisms in a concrete setting, let us first focus on TeV scale models. Here, one postulates a bulk neutrino, which is a singlet under the electroweak gauge group. Let us denote the bulk neutrino by $\nu_B(x^\mu, y)$. The bulk neutrino is represented by a four-component spinor and can be split into two chiral Weyl two-component spinors as $\nu_B^T = (\chi^T, -i\phi^\dagger\sigma_2)$. The two-component spinors χ and ϕ can be decomposed in terms of four-dimensional Fourier components as follows:

$$\chi(x, y) = \frac{1}{\sqrt{2R}}\chi_{+,0} + \frac{1}{\sqrt{R}} \sum_{n=1}^{\infty} \left(\chi_{+,n} \cos \frac{n\pi y}{R} + i\chi_{-,n} \sin \frac{n\pi y}{R} \right). \quad (95)$$

There is a similar expression for ϕ . It has a five-dimensional kinetic energy term and a coupling with the brane field $L(x^\mu)$. The full Lagrangian involving the ν_B is

$$\mathcal{L} = i\bar{\nu}_B\gamma_\mu\partial^\mu\nu_B + \kappa\bar{L}H\nu_{BR}(x, y = 0) + i\int dy\bar{\nu}_{BL}(x, y)\partial_5\nu_{BR}(x, y) + \text{h.c.}, \quad (96)$$

where H denotes the Higgs doublet, and $\kappa = hM_*/M_{Pl}$ is the suppressed Yukawa coupling. This leads to a Dirac mass for the neutrino [68] given by

$$m = \frac{h\nu_{wk}M_*}{M_{Pl}}, \quad (97)$$

where ν_{wk} is the scale of $SU(2)_L$ breaking. In terms of the two-component fields, the mass term coming from the fifth component of the kinetic energy connects the fields χ_+ with ϕ_- and χ_- with ϕ_+ , whereas it is only the ϕ_+ (or $\nu_{B,R,+}$) which couples with the brane neutrino $\nu_{e,L}$. Thus, as far as the standard model particles and forces go, the fields ϕ_- and χ_+ are totally decoupled, and we will not consider them here. The mass matrix that we will write below therefore connects only ν_{eL} , $\phi_{+,n}$ and $\chi_{-,n}$.

From equation (96), we conclude that for $M_* \sim 10$ TeV, this leads to $m \simeq 10^{-4}h$ eV. It is encouraging that this number is in the right range to be of interest in the discussion of solar neutrino oscillation if the Yukawa coupling h is appropriately chosen. Furthermore, this neutrino is mixed with all the KK modes of the bulk neutrino, with a mixing mass $\sim\sqrt{2}m$; since the n th KK mode has a mass $nR^{-1} \equiv n\mu$, the mixing angle is given by $\sqrt{2}mR/n$. Note that for $R \sim 0.1$ mm, this mixing angle is of the right order to be important in MSW transitions of solar neutrinos.

It is worth pointing out that this suppression of m is independent of the number and radius hierarchy of the extra dimensions, provided that our bulk neutrino propagates in the whole bulk. For simplicity, we will assume that there is only one extra dimension with radius of the order of a millimetre.

Secondly, the above discussion can be extended in a very straightforward manner to the case of three generations. The simplest thing to do is to add three bulk neutrinos and consider the Lagrangian to be

$$\mathcal{L} = i\bar{\nu}_{B,\alpha}\gamma_\mu\partial^\mu\nu_{B,\alpha} + \kappa_{\alpha\beta}\bar{L}_\alpha H\nu_{B\beta R}(x, y = 0) + i\int dy\bar{\nu}_{B\alpha L}(x, y)\partial_5\nu_{B\alpha R}(x, y) + \text{h.c.} \quad (98)$$

One can now diagonalize $\kappa_{\alpha\beta}$ by rotating both the bulk and the active neutrinos. The mixing matrix then becomes the neutrino mixing matrix \mathbf{U} discussed in the text. In this basis (the mass eigenstate basis), one can diagonalize the mass matrix involving the ν_i s and the bulk neutrinos to get the mixings between the active and the bulk tower. There are now three mixing parameters, one for each mass eigenstate denoted by $\xi_i \equiv \sqrt{2}m_{,i}R$ and mixing angle for each mass eigenstate to the n th KK mode of the corresponding bulk neutrinos is given by ξ/n . The observed oscillation data can then be used to put limits on ξ_i ; we discuss this in a subsequent section.

It is also worth noting that due to the presence of the infinite tower mixed with the active neutrino, the oscillation probabilities are distorted in a way which is very different from the case of oscillation to a single neutrino level. This has the implication that if at some point the complete oscillation of a neutrino (as opposed to just the overall suppression as is the case now) is observed, it will be possible to put stronger limits on the parameter ξ_i from data.

4.1.1. Neutrino propagation in matter with a bulk neutrino tower. Let us discuss the propagation of a neutrino in matter in the bulk tower neutrino models. For this purpose we have to consider the neutrino mass matrix in the flavour basis in the presence of matter effect, to be denoted by δ_{ee} . This looks as follows in the basis: $(\nu_e, \nu_{BR,+}^{(0)}, \nu_{BL,-}^{(1)}, \nu_{BR,+}^{(1)}, \nu_{BL,-}^{(2)}, \nu_{BR,+}^{(2)}, \dots)$ is given by

$$\mathcal{M} = \mathcal{M}_{\text{TeV}} \equiv \begin{pmatrix} \delta_{ee} & m & 0 & \sqrt{2}m & 0 & \sqrt{2}m & \cdot & \cdot \\ m & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \mu & 0 & 0 & \cdot & \cdot \\ \sqrt{2}m & 0 & \mu & 0 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 2\mu & \cdot & \cdot \\ \sqrt{2}m & 0 & 0 & 0 & 2\mu & 0 & \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad (99)$$

where δ_{ee} is a possible matter effect. One can evaluate the eigenvalues and the eigenstates of this matrix. The former are the solutions of the transcendental equation

$$m_n = \delta_{ee} + \frac{\pi m^2}{\mu} \cot\left(\frac{\pi m_n}{\mu}\right). \quad (100)$$

The equation for the eigenstates is

$$\tilde{\nu}_n = \frac{1}{N_n} \left[\nu_e + \frac{m}{m_n} \nu_{B,+}^{(0)} + \sum_k \sqrt{2}m \left(\frac{m_n}{m_n^2 - k^2 \mu^2} \nu_{B,-}^{(k)} + \frac{k\mu}{m_n^2 - k^2 \mu^2} \nu_{B,+}^{(k)} \right) \right], \quad (101)$$

where we have used the notation \pm for the left- and right-handed parts of the KK modes of the bulk neutrino in the two-component notation and dropped the L, R subscripts, the sum over k runs through the KK modes, and N_n is the normalization factor given by

$$N_n^2 = 1 + m^2 \pi^2 R^2 + \frac{(m_n - \delta_{ee})^2}{m^2}. \quad (102)$$

One of the very striking effects of the KK tower of bulk neutrinos is the presence of multiple MSW resonances each time the neutrino energy is such that one satisfies the resonance condition in a medium. As in the two-neutrino matter resonance, each time there is a level crossing by the ν_e due to its matter effect, there will be a dip in the survival probability of the electron neutrino.

To understand the origin of the dips, note that the MSW resonance condition is given by

$$m_{res}^2 \simeq \frac{4G_F \rho}{\sqrt{2}m_p} E_{res}. \quad (103)$$

As a result, as the neutrino energy E increases the resonance condition is satisfied for higher and higher KK modes of the bulk neutrino. The survival probability at and after the resonance is given by

$$P_{ee} \simeq \exp\left(-\frac{\pi m_n^2 \sin^2 2\theta}{2E} R_{eff}\right) \quad (104)$$

for $E \geq E_{res}$ and $P_{ee} \sim 1$ for $E < E_{res}$. The cumulative survival probability is given by $P = P^{(1)} P^{(2)} \dots P^{(n)}$. Note that due to the exponent being larger than one at the resonance (which can

be checked by putting numbers), as soon as a new KK level is crossed, the P_{ee} dips to a value much less than one and then starts to rise as E increases giving rise to the dip structure. The net effect of the combined dip structure is to flatten the spectrum.

The typical values of the survival probability within the 8B region (~ 6 to ~ 14 MeV) are quite sensitive to the value of mR . As can be seen from equation (102), higher mR increases $1/N_n \approx m/m_n \approx mR/n$ for various n , and thereby increases ν_e coupling with higher mass eigenstates, strengthens MSW resonances and lowers ν_e survival probability. Thus, searching for dips in solar neutrino spectrum is one way to probe the size of extra dimensions. The same dip phenomena could also appear in higher-energy atmospheric spectra. Since the resonance condition given by the above formula implies that the higher the energy the higher the mass difference squared (or lower the extra dimension radius) probed, search for dips in higher-energy neutrinos can, in principle, reveal the existence of extra dimensions of smaller sizes that cannot be probed by solar neutrinos.

4.2. Phenomenological and cosmological constraints on bulk neutrino models

A generic feature of understanding neutrino mass via bulk neutrinos in models with large flat extra dimension is the presence of an infinite tower of closely spaced sterile neutrinos mixed with active neutrino (say ν_e). Information about the nature of extra dimensions can therefore be obtained by looking at the phenomenological as well as cosmological effects of this mixing with the infinite tower.

4.2.1. Neutrino oscillation constraints. To see the kind of constraints one can derive, let us take a simple model where there are three bulk neutrinos (i.e. three infinite towers), each giving mass to one family of neutrinos. In this minimal model, a simple rotation of the active neutrinos takes the neutrinos to the mass basis and separates three towers. It has been shown in recent papers through detailed analysis [70], that while this minimal model can adequately explain both the solar and atmospheric neutrino oscillation phenomena essentially by arranging the active bulk neutrino mixing, it is not possible to accommodate the LSND results. Furthermore, since generic mixing of the various mass eigenstate neutrinos to the n th KK mode of the corresponding bulk neutrino goes like $m_i R/n$, if R is sizeable, the observable effect simulating a multitude of sterile neutrinos should be present. Since both present atmospheric as well as solar neutral current data from SNO severely constrain the oscillation of the active neutrinos to the sterile ones, this puts a constraint on R . For instance, considering the muon neutrino oscillation to bulk neutrinos in the atmospheric neutrino data, we can take $m_3 R \leq 10\%$. This implies that $R \leq 2 \text{ eV}^{-1}$ or R less than $40 \mu\text{m}$. This constraint is more stringent than the one derived from direct searches for deviations from the inverse square law [71].

4.2.2. Big Bang nucleosynthesis constraints. On the cosmological front, since Big Bang nucleosynthesis is a very sensitive measure of the number of extra neutrino species mixed with the active neutrinos, one should be able to get information about this class of models from these data. The basic idea here is that at the epoch of nucleosynthesis, neutrino oscillation to the bulk modes can bring in the modes into thermodynamic equilibrium with all other relativistic species. If that happens each mode will contribute an amount ρ_ν to the energy density of the universe and speed up the expansion. The faster the expansion of the universe, the earlier the weak interactions go out of equilibrium (to be called freeze-out). Since the neutron to proton ratio depends very

sensitively on the temperature of freeze-out, i.e. $n/p \sim e^{-(m_n - m_p)/T_*}$, the higher the freeze out temperature T_* , the higher the neutron fraction and hence the helium fraction of the universe.

Higher the mixing strength of the active modes with the bulk modes, the more modes from the bulk that get into thermal equilibrium with the electron. Therefore, to be consistent with observed Helium abundance, one must have a restriction on the mixing parameter $\sqrt{2}mR$.

Very careful analysis of this has been done in two papers [72] by solving the Boltzman equation for the generation of sterile neutrinos from active neutrinos at the BBN epoch. For the case of one space dimension, it was concluded in [72] that the active-bulk mixing and the inverse radius of the extra dimension μ satisfy the constraint

$$\left(\frac{\mu}{\text{eV}}\right)^{0.92} \sin^2 2\theta \leq 7.06 \times 10^{-4}. \quad (105)$$

Applying this to the tau neutrino, we find an even more stringent limit on the size of the extra dimensions R than the one just listed, i.e. $R \leq 1.5 \mu\text{m}$.

4.2.3. Enhancement of magnetic moments. Another interesting consequence of the presence of the bulk tower is in its effect on the magnetic moment of the active neutrinos. As is well known [73], if one adds a singlet right-handed neutrino to the standard model to give a Dirac mass to the neutrino, this induces a magnetic moment $\mu_\nu = 10^{-19}(m_n u/eV)\mu_B$, where μ_B is the Bohr magneton ($\mu_B = e/2m_e c$). Since in the bulk neutrino models, there is an infinite tower of neutrinos mixed with the active neutrinos, there is a $\mu_{\nu_e \nu_{B,p}} = 10^{-19}(\sqrt{2}m/eV)\mu_B$ connecting the active neutrino with each KK mode of the bulk neutrino. In a neutrino scattering process $\nu_e + e$ with neutrino energy E_ν , all bulk neutrino modes upto E_ν will be excited [74]. Thus, the effective neutrino magnetic moment will appear to be $\mu_{\text{eff}} \simeq 10^{-19}(m/eV)(ER)^{1/2}$ where R is the radius of the extra dimension. For $R \sim \text{mm}$, this enhances the magnetic moment by almost a factor of a million. Thus, this could provide an interesting way to probe the existence of extra dimensions. For example, it has recently been shown that for the case of two extra dimensions, this effect can significantly distort the spectral shape of $d\sigma/dT$ versus T (where T is the recoil energy) in $\nu_e - e$ scattering for very low T (in the 1–100 keV range) [75].

A note of caution is that this apparent enhancement is effective only when the KK modes of the bulk neutrino can be excited. For instance, when neutrino spin precesses in a magnetic field, the most of the KK modes do not get excited due to energy momentum conservation and therefore, the magnetic moment enhancement does not take place.

4.2.4. Other implications. Another interesting implication of the bulk neutrino tower appears if the brane model is not the standard model with one Higgs doublet but with two. In this case, the physical charged Higgs can decay into charged lepton and the bulk tower but with the difference that unlike in the normal two Higgs extension, the final state charged lepton emitted in this process will have left-handed helicity, whereas in models without any bulk neutrino, the final state charged lepton will have right-handed helicity [76].

The presence of the bulk neutrino tower also leads to new contributions to flavour changing leptonic rare decays such as $\mu \rightarrow e + \gamma$ etc. They in turn lead to constraints on the fundamental scale of nature.⁶

⁶ For laboratory constraints on bulk neutrino models, see e.g. [77].

4.3. Neutrino mass in low-scale gravity models without bulk neutrinos

Since the presence of light tower of bulk neutrinos is so constraining and *ad hoc* forbidding of fully allowed operators is theoretically unappealing for the brane bulk picture with the standard model in the brane, it is important to search for alternative ways to solve the neutrino mass problem. In these kinds of scenario, the strategy is to search for higher-dimensional models that will lead to the standard model after compactification of the extra dimensions and yet allow for the possibility of a low fundamental scale. One such alternative has recently been proposed by having the left–right symmetric model or an extension of the standard model with only a local $B - L$ symmetry in the bulk and looking for the standard model in the zero mode part of the spectrum [78]. It is assumed that all fermions are in the bulk. The number of extra space dimensions can either be one or two. The model with two extra space dimensions has additional symmetry which also helps to solve the proton decay problem. So we first give the example of the six-dimensional model with the gauge group $SU(2)_L \times U(1)_{I_{3R}} \times U(1)_{B-L}$. Such models have been called universal extra dimension models [80].

The minimal fermion content of this model is dictated by gravitational anomaly cancellation to be [80]

$$Q_+(2, 0, 1/3), \quad \psi_+(2, 0, -1), \quad U_-(1, 1/2, 1/3), \quad D_-(1, -1/2, 1/3), \\ E_-(1, -1/2, -1), \quad N_-(1, +1/2, -1),$$

where $Q = (u, d)$ and $\psi = (v, e)$ and \pm denote the six-dimensional chirality; the numbers in the parentheses are the gauge quantum numbers. Note that each fermion field is a four-component field with two four-dimensional two-component spinors with opposite chirality, e.g. Q has a left chiral Q_L and a right chiral field Q_R . As such the theory is vector like at this stage and we will need orbifold projections to obtain a chiral theory. We choose one Higgs doublet $\phi(2, -1/2, 0)$ and a singlet $B - L$ carrying Higgs boson $\chi(1, 1/2, -1)$. We compactify the theory on a T_2/Z_2 orbifold, where T_2 is defined by the extra coordinates $y_{1,2}$ satisfying the following conditions: $y_{1,2} = y_{1,2} + 2\pi R$ and Z_2 operates on the two extra coordinates as follows: $(y_1, y_2) \rightarrow (-y_1, -y_2)$. We now impose the orbifold conditions on the fields as follows. We choose the following fields to be even under the Z_2 symmetry: $Q_L, \psi_L, U_R, D_R, E_R, N_L$; the kinetic energy terms then force the opposite chirality states to be odd under Z_2 . Note specifically that, in contrast with the U, D, E fields, it is the N_L which is chosen even under Z_2 . This is crucial to our understanding of neutrino masses. As is well known, the even fields when Fourier expanded involve only the $\cos ny/R$ and the odd fields only $\sin ny/R$. As a result only the Z_2 even fields have zero modes. Thus, with the above compactification, below the mass scale R^{-1} , the only fermionic modes are those of the standard model plus the N_L^0 . When we give vev to the field $\langle \chi \rangle = v_{BL}$, it breaks the group down to the standard model. We will choose $v_{B-L} \sim 800$ GeV to a TeV.

Before discussing the implications of the model for neutrino masses and proton decay, let us study the extra symmetries of the four-dimensional theory implied by the fact that it derives from a six-dimensional one.

First, the discrete translational symmetry ensures the conservation of the fifth and sixth momentum components, p_a , which are quantized in integer factors of $1/R$.

Secondly, in the full uncompactified six-dimensional theory, there is an extra $U(1)_{45}$ symmetry associated with the rotations in the y_1, y_2 plane. After compactification, the $U(1)_{45}$ invariance reduces to a Z_4 symmetry. Therefore invariance under the $SO(1, 3) \times Z_4$ space–time

Table 4. $U(1)_{45}$ charges of the various fermions in the $SU(2)_L \times U(1)_{I_{3R}} \times U(1)_{B-L}$ model.

Q_L, ψ_L	+1/2
U_R, D_R, E_R, N_R	+1/2
Q_R, ψ_R, U_L	-1/2
E_L, D_L, N_L	-1/2

Lorentz transformations must be imposed on all possible operators allowed in the effective four-dimensional theory, i.e. the allowed operators will be those that are invariant under the whole $SO(1, 5)$ symmetry, plus probably those for which the sum of fermion $U(1)_{45}$ charges is equal to zero modulus 8. The reasoning is as follows: the Z_4 spatial symmetry, actually translates into a Z_8 symmetry group for the spinorial representation. In fact, under a $\pi/2$ rotation of the x_4-x_5 plane a fermion transforms as $\Psi(x') = U\Psi(x)$ with $U = \exp[i(\pi/2)\Sigma_{45}/2]$, where $\Sigma_{45} = i[\Gamma^4, \Gamma^5]/2$ is the generator of the $U(1)_{45}$ group.

To see which operators are allowed, we need to know the $U(1)_{45}$ quantum numbers of the theory which can be easily read of from the six-dimensional theory and are given in table 4.

We will use these quantum numbers below.

Let us now turn to understanding the small neutrino mass in this model. Note that in this model due to our orbifold assignments and choice of the gauge group coupled with the residual Z_8 symmetry discussed above, we only have one term that to leading order can contribute to neutrino masses and the term is $\lambda\psi_L^T C^{-1} N_L \phi(\chi^*)^2 / M_*^5$.

The following potentially dangerous terms are forbidden for various reasons in this six-dimensional theory:

- $(\psi_L \phi)^2 / M_*$ is forbidden by $B - L$ symmetry.
- Terms like $\bar{\psi}_L \phi N_R$, although allowed are also not problematic due to Z_2 quantum numbers which imply that the N_R has no zero modes.
- $(\psi_L \phi)^2 (\chi^*)^2 / M_*^7$ and $N_L N_L (\chi^*)^2$ are forbidden by the residual Z_8 symmetry.

These operators are written in the six-dimensional field theory. Upon compactification to the four-dimensional theory on our orbifold, the operator that leads to neutrino mass has the form $\lambda\psi_L^{(0)T} C^{-1} N_L^{(0)} \phi^{(0)} (\chi^{(0)*})^2 / M_*^5 R^3$. Using $M_* \simeq 100 \text{ TeV}$ and $R^{-1} \sim \text{TeV}$ and using $\lambda \sim 0.1$, we find for that it leads to $m_\nu \sim \text{eV}$, which is in the right range without any fine tuning. Furthermore, the neutrinos in this model are Dirac particles since all Majorana terms are forbidden to leading order by the Z_8 symmetry.

It is straightforward to have a left–right symmetric embedding of the local $B - L$ symmetry and show in a five-dimensional example how it can lead to small neutrino masses despite the fact that the fundamental scale of the model is in the TeV range. These considerations are easily extended to the six space–time dimensions.

This model does not have a tower of bulk neutrinos with mass gap of meV type; secondly, there is no need to invoke global $B - L$ symmetry to prevent undesirable terms. This argument also carries over to the 6-D extension of this model, which one may want for the purpose of suppressing proton decay.

Conclusions and outlook

At the moment, neutrino oscillation experiments have provided the first evidence for new physics beyond the standard model. The field of neutrino physics therefore has become central to the study of new physics at the TeV scale and beyond. The other area which most theorists believe will be the next one to emerge from experiments is supersymmetry. We have therefore assumed supersymmetry in most of our discussions, although in the last section, we consider low-scale extra dimensional models without supersymmetry.

What have we learned so far? One point that seems very clear is that there is probably a set of three right-handed neutrinos which restore quark lepton symmetry to physics; secondly there must be a local $B - L$ symmetry at some high scale beyond the standard model that is responsible for the RH neutrinos being so far below the Planck scale. While there are very appealing arguments that the scale of $B - L$ symmetry is close to 10^{14} – 10^{16} GeVs, in models with extra dimensions, one cannot rule out the possibility that it is around a few TeV. The third point that one may suspect is that the right-handed neutrino spectrum may be split into a heavier one and two others which are nearby. If this suspicion is confirmed, that would point towards an $SU(2)_H$ horizontal symmetry or perhaps even an $SU(3)_H$ symmetry which breaks into an $SU(2)_H$ symmetry (although simple anomaly considerations prefer the first alternative).

The correct theory should explain:

- (i) Why are both the solar and atmospheric mixing angles nearly maximal? In particular, is there any significance to the observation that $\theta_{12}^v + \theta_{12}^q = \theta_{23}^v + \theta_{23}^q \simeq \pi/4$ [81]?
- (ii) Why is $\Delta m_{\odot}^2 \ll \Delta m_A^2$ but $\Delta m_{\odot}^2 / \Delta m_A^2 \gg m_{\mu}^2 / m_{\tau}^2$ as would be expected in a hierarchical scenario?
- (iii) What is responsible for the smallness of U_{e3} ?
- (iv) What is the nature of CP phases in the lepton sector and what is their relation to the CP phases possibly responsible for baryogenesis via leptogenesis?
- (v) What is the complete mass spectrum for neutrinos?

As we move into the era of precision neutrino measurement science (PNMS), these and other questions will provide an exciting venue for research in neutrino physics.

Acknowledgments

This work is supported by the National Science Foundation grant number PHY-0099544. Parts of the work were presented at an ICTP summer school on neutrinos. I thank Alexei Smirnov for the invitation and kind hospitality at ICTP during the school, and K S Babu, B Dutta, A Smirnov, G Senjanović, A Perez-Lorenzana, B Kayser, G Fuller and E Akhmedov for discussions.

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