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A Quantum Field Theoretical Treatment of Neutrino Oscillations

Neutrinos Propagating through Vacuum and Matter

Diego Pallin

Master's Thesis

Division of Mathematical Physics
Department of Physics
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Abstract

This thesis is divided into 5 chapters. Chapter 1 contains an introduction to the subject. Chapter 2 contains a general review of neutrino physics, starting with a short introduction of the standard model. It then reviews neutrino oscillations, different models for massive neutrinos and concludes by presenting some experiments involving neutrinos. The two main parts are neutrino oscillations and a discussion of the Dirac and Majorana character of neutrinos.

In Chapter 3, there is first a general review of basic Quantum Field Theory (QFT) and an explanation of the notation used in the following chapters. The main part of the chapter is devoted to the derivation of an expression for the oscillation probability using QFT. We start out by using the Euler-Lagrange equation on that part of the Lagrangian density, from the standard electroweak theory, that is of interest to us. We arrive at an expression for the oscillation probability by using a wave packet approach and the Feynman fermion propagator to connect the initial and final states.

In chapter 4, we treat neutrino oscillations in matter, first by a general discussion. We then derive an expression for the oscillation probability when neutrinos are propagating through a static uniform background, within the framework of QFT, this is the simplest case of neutrinos interacting with the background.

In chapter 5, we compare the expression for the oscillation probability in vacuum, which we derived in chapter 3, with other expressions derived using Quantum Mechanics (QM) and QFT. We come to the conclusion that the QFT treatment we used, in the relativistic limit, is similar to a QM treatment using wave packets.

Key words: QFT, QM, neutrino, neutrino oscillations, neutrino flavor, Dirac and Majorana neutrinos, matter effects, Gaussian wave packet, Feynman fermion propagator.

Preface

This thesis is the result of work done at the Division of Mathematical Physics, Department of Physics at KTH between September 2001 and April 2002. In order to gain insight into the area of neutrino physics, I first started out by reading different books on the subject. In parallel with that I also read books and took a course on QFT. The quite long introduction period was necessary in order to obtain enough knowledge to be able to start the real work.

The thesis starts with a short introduction to the wide area of neutrino physics. In chapter two I give a general review of neutrino physics. After a short summary of different neutrino properties I then give a short introduction to the standard model and why neutrinos are assumed to be massless. One major part is about neutrino oscillations, starting with the derivation of the QM probability oscillation formula. I then discuss different aspects of the unitary mixing matrix and what consequences $\mathcal{CPT}$ and Lorentz violations have for the transition probability. There is also an extensive discussion about massive neutrinos where I study different aspects, symmetry properties and different mass terms, of Dirac and Majorana neutrinos. I also mention some different mass models such as Grand Unified Theories (GUTs) and review some different experiments.

In chapter 3, I start by reviewing some basic QFT. This is needed in the rest of the chapter, which is also the main part of the thesis. In order to arrive at an expression for the transition probability, I start with the part of the Lagrangian density which is of interest to us. By using the Euler-Lagrange equations of motion, I then derive an equation of motion for the different neutrino mass fields. In order to solve these coupled Partial Differential Equations (PDEs), I used perturbation theory and expanded the different neutrino mass fields in terms of creation and annihilation operators. It was then possible to write down an expression for the flavor states in terms of the mass eigenfields in a wave packet approach. The initial and final states are connected by the Feynman fermion propagator, which arise naturally in the expression for the amplitude as an expectation value of the time ordered product of mass eigenfields. The expression for the transition amplitude contained several integrals and it took some time to find the way in which order to perform them. I finally assumed that the distribution of mass eigenfields was Gaussian. This allowed me to write down an analytic expression for the transition probability in the relativistic limit.

Chapter 4 treats neutrinos propagating through matter and I discuss why different neutrino flavors are not affected in the same way. This question brings up the concept of neutral- and charged-current weak interactions. There are also some comments about what $\mathcal{CP}$ symmetries imply for the dispersion relations for neutrinos and antineutrinos. I then derive an expression for the oscillation probability in the case when the neutrinos are propagating through a static uniform background. The key problem is to find the propagator. The correct way would be to solve a system of PDEs for the neutrino fields and then compute the time ordered expectation value, unfortunately this is too difficult. I therefore just take into account the first non-trivial contribution, i.e. when the neutrinos interact only once with matter. The correct way would be to sum up all different contributions. I can therefore not expect to recover the QM result. Finally, I mention the QM results using both a plane wave and a wave packet approach.

In chapter 5, I review some different expressions for the transition probability derived in QM and QFT. I also compare these expressions to the one derived in chapter 3, describing the similarities and the differences.

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Contents

Chapter 1

Introduction

The neutrinos which today play an important role in different branches such as subatomic physics, astrophysics and cosmology were introduced more than 70 years ago by Wolfgang Pauli. Pauli introduced the new particle "neutron" ν to be able to describe the continuous spectrum of nuclear β decay $(A, Z) \rightarrow (A, Z+1) + e^- + \nu$. His own words regarding the mass of ν was "the mass of the neutron should be of the same order as the electron mass". In 1934, four years after the introduction of the ν, it received its final name neutrino by Fermi. In 1957, Landau, Lee and Yang, and Salam formulated the first theoretical argument in favor of a vanishing neutrino mass. The same year Goldhaber et al. measured the neutrino helicity in a celebrated experiment. The question was then, are neutrinos massive or not. One possible way to observe this was suggested by Pontecorvo [26] in 1957. It was based on the fact that if the neutrino flavors were a superposition of massive eigenstates, then the neutrinos could oscillate between these different flavors.

There are several experiments looking for neutrino oscillations: solar neutrino measurements, atmospheric neutrino measurements, reactor experiments, and accelerator experiments. Several experiments have claimed that their measurements show some evidence for neutrino oscillations. The final breakthrough came in June 1998 when the Super-Kamiokande collaboration reported to have strong evidence for neutrino oscillations. The measurements on the depletion of atmospheric muon neutrino flux fit well to a two flavor $\nu_\mu \leftrightarrow \nu_\tau$ oscillation model.

In the standard QM treatment of neutrino oscillations, the mass eigenstates are assumed to be relativistic and to have the same momentum and thus different energies. The familiar QM model describing the flavor mixing process has several conceptual difficulties compared to QFT models, see [2, 6, 12, 14, 27]. For example energy momentum conservation in the production and detection processes that forces neutrinos to be in a mass eigenstate is incompatible with neutrino oscillations. The neutrino oscillation probability is independent of the details concerning the production and detection processes only in the extremely relativistic limit. Hence, in the case that some of the mass eigenstates cannot be considered to be extremely relativistic, one has to use QFT. Of course, the QFT expression must reproduce the QM oscillation probability in the extremely relativistic limit. A very detailed review regarding different aspects and questions of neutrino oscillations in QFT can be found in [2]. Another interesting question which is important in both a QM or QFT treatment is if there exists a Fock space for the flavor eigenstates, since there exists a Fock space for the mass eigenstates. This interesting question has been discussed by Giunti et al. [11]. Fuji, Habe, and Yabuki [8] give arguments that a Fock space of flavor neutrinos does not exist.

When neutrinos propagate through matter the oscillation behavior may be affected significantly, as was pointed out by Wolfenstein [28] in 1978. This is due to the fact that in presence of matter the effective mass induced by the forward scattering of neutrinos by the background changes the flavor oscillating parameters. A QM treatment of neutrinos interacting with matter can be found in almost all books treating neutrinos. Peltoniemi et al. [23] have studied neutrino propagation in matter using a wave packet approach. Cardall and Chung [7] treats neutrino oscillations in a static uniform background by QFT and show that they recover the QM oscillation amplitude in the relativistic limit.

Chapter 2

Neutrino Physics

2.1 What are Neutrinos?

Neutrino observations have played a key role in the understanding of weak interactions in the standard model (SM). It is also believed that neutrinos may hold the key to understand physics beyond the SM. Next to radiation the neutrino is the most abundant form of matter in the universe. Thus the neutrino is an important particle.

Some properties of neutrinos: They have spin 1/2 and are believed to have no electric charge. But one of the major questions in particle physics today is: Do they have mass?

At present we know that there exists three different kinds of neutrinos ν_e, ν_μ and ν_τ. Some experiments are trying to find out if there exists a fourth type of neutrino, since that would imply the existence of a fourth type of "matter". The different neutrinos are grouped, into different generations, with the quarks and leptons in the following way:

Generation	Particles
1	u, d, e, ν_e
2	c, s, μ, ν_μ
3	t, b, τ, ν_τ

All laboratory experiments up to today show that if neutrinos really have mass it is much smaller than any of the other masses of quarks and leptons in the same generation.

For the moment there are still some unanswered questions. Besides the mass question some of these are: How many neutrino types are there? Are the neutrinos Dirac particles or Majorana particles? There are several good reviews of neutrino physics [17, 18, 20].

2.2 The Standard Model

The SM of electroweak and strong interactions is based on the gauge symmetry group $\mathrm{SU}(3)_C \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$. It describes all fundamental forces in nature except gravitation and the particles are classified into fermions and bosons in the following way:

Fermions $s = 1/2, 3/2, \ldots$		Bosons $s = 0, 1, \ldots$	
Baryons	Leptons	Mesons	Gauge particles
$p, \Lambda, \Xi, \Omega, \ldots$	$e, \mu, \tau, \nu_e, \ldots$	$\pi, K, \phi, \ldots$	$\gamma, W, Z^0, \ldots$

Here $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$ is the gauge group for the electroweak interaction called the Glashow-Weinberg-Salam (GWS) electroweak model, whereas $\mathrm{SU}(3)_C$ is the gauge group for the strong interaction known as Quantum Chromodynamics (QCD). Particles with strong interaction are called hadrons, which includes baryons and mesons. The hadrons are built up by elementary particles called quarks.

All the interactions in the SM of elementary particles are given by the Lagrangian

$$\mathcal{L}_{\mathrm{SM}} = \mathcal{L}_{\mathrm{kinetic}} + \mathcal{L}_{\mathrm{gauge}} + \mathcal{L}_{\mathrm{Yukawa}} + \mathcal{L}_{\mathrm{Higgs}} \tag{2.1}$$

where

$$\begin{aligned}
\mathcal{L}_{\mathrm{kinetic}} &= \sum_{a=1}^{3} \overline{\mathrm{Q}}_a i\not{D}\mathrm{Q}_a + \sum_{a=1}^{3} \overline{\mathrm{U}}_a i\not{D}\mathrm{U}_a + \sum_{a=1}^{3} \overline{\mathrm{D}}_a i\not{D}\mathrm{D}_a \\
&\quad + \sum_{a=1}^{3} \overline{\mathrm{L}}_a i\not{D}\mathrm{L}_a + \sum_{a=1}^{3} \overline{\mathrm{E}}_a i\not{D}\mathrm{E}_a, && (2.2)\\
\mathcal{L}_{\mathrm{gauge}} &= -\frac{1}{4}G^{\mu\nu,a}G^a_{\mu\nu} - \frac{1}{4}W^{\mu\nu,a}W^a_{\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu}, && (2.3)\\
\mathcal{L}_{\mathrm{Yukawa}} &= -\sum_{a=1}^{3}\sum_{b=1}^{3} G^U_{ab}\overline{\mathrm{Q}}_a\Phi^c\mathrm{U}_b - \sum_{a=1}^{3}\sum_{b=1}^{3} G^D_{ab}\overline{\mathrm{Q}}_a\Phi\mathrm{D}_b \\
&\quad - \sum_{a=1}^{3}\sum_{b=1}^{3} G^L_{ab}\overline{\mathrm{L}}_a\Phi\mathrm{E}_b + \mathrm{h.c.}, && (2.4)\\
\mathcal{L}_{\mathrm{Higgs}} &= (D^\mu\Phi)^\dagger(D_\mu\Phi) + \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 && (2.5)
\end{aligned}$$

and the total number of independent parameters is 18. The fermions are divided into quarks and leptons, and we have defined Q, U, D, L, and E in the following way:

	Generations		
	1	2	3
Q	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$
U	u_R	c_R	t_R
D	d_R	s_R	b_R
L	$\begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu L} \\ \mu_L^- \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau L} \\ \tau_L^- \end{pmatrix}$
E	e_R^-	μ_R^-	τ_R^-

The field strength tensors $G^{\mu\nu,a}$, $W^{\mu\nu,a}$, and $B^{\mu\nu}$ are defined in terms of coupling constants, structure constants, and the bosonic fields: The gluon g ($G^{\mu,a}$), the electroweak gauge bosons $W^{\pm}, Z^0$ ($W^{\mu,a}$), the photon γ (B^{μ}), and the Higgs H^0 (Φ). The covariant derivatives D_μ are also defined in terms of the fields B_μ etc. This presentation is very short and a more detailed discussion of the standard model can be found in [21].

Many experiments have been carried out in order to test the SM. For the moment the Higgs boson H^0 has not yet been observed. The tau neutrino was observed very recently, in 2000 at Fermilab.

When discussing neutrinos we can skip the $\mathrm{SU}(3)_C$ part since neutrinos do not have strong interactions, although of course the strong interaction enters in higher order corrections. The Hamiltonian of the weak interaction can written as

$$\mathcal{H}_{\text{weak}} = \frac{4G_F}{\sqrt{2}} \left[J^\mu(x) J_\mu^\dagger(x) + \rho K^\mu(x) K_\mu(x) \right], \tag{2.6}$$

where

$$J_\mu(x) = \overline{\mathbf{u}} \gamma_\mu P_L V \mathbf{d} + \overline{\nu} \gamma_\mu P_L \mathbf{l} \tag{2.7}$$

and

$$\begin{aligned} K_\mu(x) &= \sum_q \left[\epsilon_L(q) \overline{q} \gamma_\mu P_L q + \epsilon_R(q) \overline{q} \gamma_\mu P_R q \right] \\ &+ \frac{1}{2} \sum_\nu \overline{\nu} \gamma_\nu P_L \nu + \frac{1}{2} \sum_l \overline{l} \gamma_\mu \left[\mathbb{1} g_V(l) - \gamma_5 g_A(l) \right] l. \end{aligned} \tag{2.8}$$

Here $\epsilon_L(q)$, $\epsilon_R(q)$, $g_V(l)$ and $g_A(l)$ are expressed in terms of the Weinberg angle θ_W, see page 26 in [20]. The matrix V denotes the mixing matrix between different quark generations. The fermion fields are

$$\mathbf{u} = \begin{pmatrix} u \\ c \\ t \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad \boldsymbol{\nu} = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}, \quad \mathbf{l} = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}, \tag{2.9}$$

and the chirality projection operators are defined as

$$P_L = \frac{1}{2}(\mathbb{1} - \gamma_5), \quad P_R = \frac{1}{2}(\mathbb{1} + \gamma_5), \tag{2.10}$$

which satisfy

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0, \quad P_L + P_R = 1,$$
$$P_L^\dagger = P_L, \quad P_R^\dagger = P_R, \quad P_L \gamma^\mu = \gamma^\mu P_R. \tag{2.11}$$

2.2.1 Neutrino Mass in the Standard Model

Only one helicity state of the neutrino is present per generation in the SM. Since a Dirac mass term requires both helicity states this implies that the neutrino cannot have a Dirac mass. On the other hand, a Majorana mass term, requires only one helicity state, since it uses the opposite helicity state of the antiparticle, could still be possible. The standard model conserves lepton numbers separately for each generation. Due to the fact that a Majorana mass term would break the lepton numbers by two units, such a mass term is not possible. Thus according to the SM, neutrinos can have neither a Dirac nor Majorana mass term in any order of perturbation theory or in the presence of non-perturbative effects. The conclusion is therefore that the neutrino mass vanishes in the SM. This has some implications, for example vanishing mixings and magnetic moment.

2.3 Neutrino Oscillations

2.3.1 History

Neutrino oscillation, which is a consequence of neutrino mixing, was first suggested by Pontecorvo [26] in 1957. In 1962, Maki, Nakagawa, and Sakta [19] introduced the concept of neutrino mixing with two flavors and the confirmation of the existence of different neutrino flavors ν_e and ν_μ came in the same year. Gribov and Pontecorvo [13] developed the first theory of two neutrino oscillations and the oscillation probabilities as they are used today were formulated by Bilenky and Pontecorvo [3].

2.3.2 Quantum Mechanical Treatment

The neutrinos are in almost all experiments produced by charged-current weak interactions. In a neutrino beam, the neutrino ψ_α is created along with the antilepton $\bar{l}_\alpha$, ψ_α is not a physical particle, but a superposition of physical fields ν_j with mass m_j. This can be written as

$$\psi_\alpha = \sum_j U_{\alpha j} \nu_j \tag{2.12}$$

where we have introduced the mixing matrix U. If we take the Hermitian conjugate of Eq. (2.12) and act with it on the vacuum state $|0\rangle$ we obtain

$$|\psi_\alpha\rangle = \sum_j U_{\alpha j}^* |\nu_j\rangle \tag{2.13}$$

Introduce the flavor state basis $\mathcal{H}_{\text{flavor}} \equiv \{|\psi_\alpha\rangle\}$ and the mass eigenstate basis $\mathcal{H}_{\text{mass}} \equiv \{|\nu_j\rangle\}$. A neutrino flavor state is a superposition of different mass eigenstates and since the different mass eigenstates does not evolve in the same way as the neutrino propagates the probability of finding a certain flavor varies with time.

The simplest approach is obtained if we assume that all the neutrinos have the same momentum $\mathbf{p}$. As a consequence the different mass eigenstates $|\nu_j\rangle$ will have different energies since $E_j = \sqrt{\mathbf{p}^2 + m_j^2}$ because the masses are different. The time evolution of $|\nu_j\rangle$ in QM is given by the evolution operator e^{-iE_jt}. Using this gives $|\nu_j(t)\rangle = e^{-iE_jt}|\nu_j\rangle$. Inserting the time evolution into Eq. (2.13) we have

$$|\psi_\alpha(t)\rangle = \sum_j e^{-iE_jt}U^*_{\alpha j}|\nu_j\rangle. \tag{2.14}$$

The quantum mechanical transition amplitude $A_{\alpha\beta}(t)$ for ψ_α to be converted into ψ_β after time t is

$$\begin{aligned} A_{\alpha\beta}(t) = \langle\psi_\beta|\psi_\alpha(t)\rangle &= \sum_{j,k}\langle\nu_k|U_{\beta k}e^{-iE_jt}U^*_{\alpha j}|\nu_j\rangle \\ &= \sum_j e^{-iE_jt}U_{\beta j}U^*_{\alpha j}, \end{aligned} \tag{2.15}$$

where we have used that $\langle\nu_k|\nu_j\rangle = \delta_{jk}$, i.e. $\mathcal{H}_{\text{mass}}$ is an orthonormal basis. The probability $P_{\alpha\beta}(t)$ that the transition $\alpha \to \beta$ takes place is given by the absolute value squared of the transition amplitude. We thus have

$$\begin{aligned} P_{\alpha\beta}(t) = |A_{\alpha\beta}(t)|^2 &= |\langle\psi_\beta|\psi_\alpha(t)\rangle|^2 \\ &= \sum_{j,k} U_{\beta j}U^*_{\alpha j}U^*_{\beta k}U_{\alpha k}e^{-i(E_j-E_k)t}. \end{aligned} \tag{2.16}$$

In the relativistic limit, we can make the approximation $E_j \approx |\mathbf{p}| + \frac{m_j^2}{2|\mathbf{p}|}$ and also replace t by the distance x. This gives

$$P_{\alpha\beta}(x) = \sum_{j,k} U_{\beta j}U^*_{\alpha j}U^*_{\beta k}U_{\alpha k}e^{-i\frac{(m_j^2-m_k^2)}{2|\mathbf{p}|}x}. \tag{2.17}$$

This expression can be rewritten in many useful ways which are suitable in different situations. One example is

$$P_{\alpha\beta}(x) = \sum_{j,k} |U_{\beta j}U^*_{\alpha j}U^*_{\beta k}U_{\alpha k}| \cos\left(\frac{2\pi x}{L_{jk}} - \phi_{\alpha\beta jk}\right), \tag{2.18}$$

where we have introduced

$$L_{jk} \equiv \frac{4\pi|\mathbf{p}|}{\Delta_{jk}}, \quad \Delta_{jk} \equiv m_j^2 - m_k^2, \quad \phi_{\alpha\beta jk} \equiv \arg\left(U_{\beta j}U^*_{\alpha j}U^*_{\beta k}U_{\alpha k}\right). \tag{2.19}$$

Here L_{jk} are called oscillation lengths and gives the distance over which one has appreciable oscillation effects. These derivations can be found in all books treating neutrino oscillations. A more advanced derivation is presented in [18, 21].

If one instead assumes that all the neutrinos have the same energy E, then the different mass eigenstates $|\nu_j\rangle$ will have different momenta $|\mathbf{p}_j| = \sqrt{E^2 - m_j^2} \approx E - m_j^2/(2E)$. The transition probability can then be written as

$$P_{\alpha\beta}(x) = \sum_{j,k} U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} e^{-i\frac{(m_j^2 - m_k^2)}{2E}x}. \tag{2.20}$$

In the relativistic limit, in which $E \approx |\mathbf{p}|$, Eq. (2.20) reads

$$P_{\alpha\beta}(x) = \sum_{j,k} U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} e^{-i\frac{(m_j^2 - m_k^2)}{2|\mathbf{p}|}x}. \tag{2.21}$$

Thus, an interesting fact is that the expression for the QM transition probability in the relativistic limit is the same regardless whether we assume that the neutrinos have the same energy or the same momentum. This is discussed in [15].

2.3.3 The Unitary Mixing Matrix

The matrix U is a unitary $n \times n$ matrix when there are n neutrino flavors. This matrix U is characterized by n^2 real parameters out of which $n(n-1)/2$ are Euler angles (mixing angles) and $n(n+1)/2$ are phase factors. Not all of the phases are physically measurable.

For Dirac neutrinos the number of physically measurable phases are $n_D = (n-1)(n-2)/2$, while for Majorana neutrinos the number is $n_M = n(n-1)/2$. Thus, the difference is $n_M - n_D = n - 1$.

The probability $P_{\alpha\beta}$ depends on $n(n-1)$ parameters out of which $n-1$ are mass squared differences, $n(n-1)/2$ are mixing angles and $(n-1)(n-2)/2$ are $\mathcal{CP}$ phases. The probability $P_{\alpha\beta}$ in both the Dirac and Majorana cases may depend only on phases in the matrix U that cannot be absorbed by the phase transformation $U_{\alpha j} \to U'_{\alpha j} = e^{i\phi_\alpha} U_{\alpha j} e^{-i\phi_j}$, where $\phi_\alpha, \phi_j \in \mathbb{R}$. There are $(n-1)(n-2)/2$ such phases. This implies that it is impossible to determine if neutrinos are Dirac or Majorana particles by studying neutrino oscillations in vacuum or matter. This has been discussed in [21].

If we assume that there is oscillation only between two Dirac neutrinos, then the matrix U can be written in the following simple form

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.22}$$

This gives

$$P_{\text{conversion}}(x) = \sin^2 2\theta \sin^2\left(\frac{\Delta}{4E}x\right) \tag{2.23}$$

and

$$P_{\text{survival}}(x) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta}{4E}x\right). \tag{2.24}$$

If we instead assume oscillation only between two generations of Majorana neutrinos, then the most general form for the mixing matrix is

$$U = \begin{pmatrix} \cos\theta & e^{-i\rho}\sin\theta \\ -e^{i\rho}\sin\theta & \cos\theta \end{pmatrix}. \tag{2.25}$$

One can show that the phase ρ does not appear in the oscillation probabilities and that they are the same as for two Dirac neutrinos, i.e., Eqs. (2.23) and (2.24).

2.3.4 $\mathcal{CPT}$ and Lorentz Violations

Symmetry Properties

We are now going to discuss the different properties of the $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ transformations.

The charge conjugation operator $\mathcal{C}$ transforms a neutrino state $|\nu(\mathbf{p}, s)\rangle$ with momentum $\mathbf{p}$ and spin s into an antineutrino state $|\overline{\nu}(\mathbf{p}, s)\rangle$ according to

$$\mathcal{C}|\nu(\mathbf{p}, s)\rangle = \eta_C|\overline{\nu}(\mathbf{p}, s)\rangle, \tag{2.26}$$

where η_C is a phase factor. In a second quantized theory $\mathcal{C}$ is a unitary operator. We define the charge conjugate neutrino field ν^C as $\nu^C \equiv \mathcal{C}\nu\mathcal{C}^\dagger$. For a more detailed treatment see pages 18-19 in [18].

The parity operator $\mathcal{P}$, which is unitary, transforms $\mathbf{r}$ into $-\mathbf{r}$. Using this operator, gives

$$\mathcal{P}|\nu(\mathbf{p}, s)\rangle = \eta_P|\nu(-\mathbf{p}, s)\rangle, \tag{2.27}$$

where the phase factor η_P is called the intrinsic parity.

The time reversal operator $\mathcal{T}$ changes t to $-t$, but keeps $\mathbf{r}$ invariant. This means that

$$\mathcal{T}|\nu(\mathbf{p}, s)\rangle = \eta_T^s|\nu(-\mathbf{p}, -s)\rangle, \tag{2.28}$$

where the phase factor η_T^s depends on the spin s. The operator $\mathcal{T}$ has a special property namely that it is antiunitary.

We also define the operator $\Theta = \mathcal{CPT}$, which is antiunitary, since $\mathcal{T}$ is, and the operator $\Sigma = \mathcal{CP}$.

Consequences of $\mathcal{CPT}$ and Lorentz Violations

It is possible that $\mathcal{CPT}$ and Lorentz symmetries are violated at very small length scales. Neutrino oscillations can be used to put constraints on the symmetry breaking parameters, because neutrino oscillations are very sensitive to $\mathcal{CPT}$ and Lorentz violations. $\mathcal{CP}$, $\mathcal{T}$, and $\mathcal{CPT}$ violations has the following consequences for neutrino oscillations, (assume that $\beta \neq \alpha$)

- $\mathcal{CP}$ is violated $\Rightarrow P_{\alpha\beta} \neq P_{\overline{\alpha}\overline{\beta}}$.
- $\mathcal{T}$-invariance is violated $\Rightarrow P_{\alpha\beta} \neq P_{\beta\alpha}$.
- $\mathcal{CPT}$ is violated $\Rightarrow P_{\alpha\beta} \neq P_{\overline{\beta}\overline{\alpha}}$ and $P_{\alpha\alpha} \neq P_{\overline{\alpha}\overline{\alpha}}$.

The modification of the SM to describe small departures from exact Lorentz invariance has been developed by Colladay and Kostelecky. References and further information can be found in a review by Pakvasa [22].

2.4 Massive Neutrinos

The question of a plausible neutrino mass cannot be discussed in isolation since, it is related to other issues of which the most important is that of neutrino mixing. But if mixing occurs lepton numbers cannot remain as valid global symmetries.

2.4.1 Experimental Limits

In order to get some limits on the masses one can use kinematical considerations on different types of reactions. By studying the electron spectrum of the β-decay,

$$n \rightarrow p + e + \overline{\nu}_e, \tag{2.29}$$

it is possible to establish an upper bound for the ν_e mass. Upper bounds for the ν_μ and ν_τ masses are obtained by treating the following reactions,

$$\pi^+ \rightarrow \mu^+ + \nu_\mu \tag{2.30}$$

and

$$\tau \rightarrow 2\pi^+ + 3\pi^- + \pi^0 + \nu_\tau. \tag{2.31}$$

Experimentally we have the following upper limits on the neutrino masses [15]

$$\begin{aligned} m_{\nu_e} &< 3 \text{ eV}, \\ m_{\nu_\mu} &< 190 \text{ keV}, \\ m_{\nu_\tau} &< 18.2 \text{ MeV}. \end{aligned} \tag{2.32}$$

Only the mass eigenstates have well defined masses and not the flavor eigenstates. Thus the values given above are only valid when there is no mixing at all, i.e. when the flavor eigenstates are equal to the mass eigenstates.

2.4.2 Dirac or Majorana Neutrinos

If we assume that neutrinos are massive we are still faced with the question whether they are Dirac or Majorana particles. If the neutrinos are Majorana particles, then they are their own antiparticles, since they do not have any electric charge. But they could also be Dirac particles like all charged leptons and quarks have to be as a consequence of electric charge conservation. It is impossible to determine if neutrinos are Dirac or Majorana particles by studying neutrino oscillations in vacuum or matter as stated earlier. This is sometimes also called the "Dirac-Majorana confusion theorem" [25].

When discussing the differences between Dirac and Majorana neutrinos we will introduce the notation ν_D and ν_M to denote Dirac and Majorana neutrinos, respectively.

Dirac Neutrinos

The left-handed and right-handed Weyl fields ν_L and ν_R, also called chiral fields, are defined by

$$\nu_L \equiv P_L \nu = \frac{\mathbb{1} - \gamma_5}{2}\nu, \quad \nu_R \equiv P_R \nu = \frac{\mathbb{1} + \gamma_5}{2}\nu. \tag{2.33}$$

Since $\gamma_5 \nu_L = -\nu_L$ and $\gamma_5 \nu_R = \nu_R$, the chiral fields are eigenfields of γ_5. This is valid both for massless and massive neutrinos. In the case of massless neutrinos, ν_L and ν_R are eigenfields of the helicity projection operator, $\boldsymbol{\sigma} \cdot \mathbf{p}$, i.e.,

$$\boldsymbol{\sigma} \cdot \mathbf{p} \nu_L = -\nu_L, \quad \boldsymbol{\sigma} \cdot \mathbf{p} \nu_R = \nu_R. \tag{2.34}$$

The helicity operator projects out states with the spin either along or against the direction of motion. For massive neutrinos the helicity states are physical, whereas the chiral states are not. However, the chiral states can be written as a linear combination of helicity eigenstates.

For a Dirac neutrino there are two fields which can be used in the construction of a Dirac mass term, namely ν_L and ν_R. The SM does not include ν_R, since it has not been observed in Nature.

Majorana Neutrinos

The case of a Majorana particle was first discussed by Majorana in 1937.

A Majorana particle has half as many degrees of freedom as a Dirac particle. The plane wave expansion of a Majorana field operator is

$$\nu_M(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E(\mathbf{p})}} \sum_s \left(a^s(\mathbf{p}) u^s(p) e^{-ipx} + \lambda a^{s\dagger}(\mathbf{p}) v^s(p) e^{ipx} \right), \tag{2.35}$$

where the meaning of λ is that the antiparticle is the same as the corresponding particle except for a phase λ. The Lagrangian density for a free Majorana field is $\mathcal{L}(x) = \frac{1}{2}\overline{\psi}(x)(i\partial\!\!\!/ - m)\psi(x)$. One can, in addition, show that the propagator of a Majorana field is given by the same expression as that of a Dirac field. However, the Majorana propagator $\langle 0|T(\nu_M(y')\overline{\nu_M}(x'))|0\rangle$ describes $\overline{\nu} \to \nu$ or $\nu \to \overline{\nu}$ propagation depending on the time ordering, whereas the Dirac propagator describes $\nu \to \nu$ or $\overline{\nu} \to \overline{\nu}$ propagation.

A Majorana neutrino ν_M is defined as a particle that has the following property

$$\nu_M = \nu_M^C \equiv \mathcal{C}\nu_M\mathcal{C}^\dagger. \tag{2.36}$$

Since $\nu_M = \nu_M^C$, it has interesting $\mathcal{CP}$ and $\mathcal{CPT}$ properties. For Majorana neutrinos the $\mathcal{CPT}$ properties are the same for both free and interacting particles, while the $\mathcal{C}$ and $\mathcal{CP}$ properties are only valid for free particles. A Majorana neutrino is an eigenstate of $\mathcal{CP}$ only if we neglect the fact that the weak interaction violates $\mathcal{CP}$. In the case of a free Majorana neutrino, we can write

$$\Sigma|\nu_M(\mathbf{p},s)\rangle = \eta_\Sigma^M|\nu_M(-\mathbf{p},s)\rangle. \tag{2.37}$$

A direct calculation gives that $\eta_\Sigma^M = -\left(\eta_\Sigma^M\right)^*$, showing that the $\mathcal{CP}$ phase of a Majorana neutrino is purely imaginary. A very important property is that two different Majorana neutrinos can have opposite $\mathcal{CP}$ phases. This plays an important role in the double β decay, see for example [17]. In order to make a Dirac field one needs two Majorana fields with the same mass, but with different $\mathcal{CP}$ phases.

2.4.3 Different Models for Neutrino Masses

Here we will discuss different models for one generation of neutrinos; the generalization to n flavors can be found in [18, 20, 21].

Dirac Mass Term

From the Lagrangian density for a Dirac field $\mathcal{L} = \overline{\nu}(i\partial\!\!\!/ - m)\nu$ we identify the Dirac mass term $\mathcal{L}_D = -m\overline{\nu}\nu$. Using $\nu = \nu_L + \nu_R$, we have $\mathcal{L}_D = -m(\overline{\nu_L}\nu_R + \overline{\nu_R}\nu_L)$, since $\overline{\nu_L}\nu_L$ and $\overline{\nu_R}\nu_R$ vanish, see pages 68-69 in [21].

Majorana Mass Term

The combinations $\overline{\nu^C}\nu^C$, $\overline{\nu^C}\nu$ and $\overline{\nu}\nu^C$ are all Lorentz invariant expressions. Inserting $\nu = \nu_L + \nu_R$, one obtains the following non-vanishing terms,

$$\overline{\nu_L^C}\nu_L, \quad \overline{\nu_L}\nu_L^C, \quad \overline{\nu_R^C}\nu_R, \quad \overline{\nu_R}\nu_R^C. \tag{2.38}$$

Note that we have introduced the following notation $\nu_{L,R}^C \equiv (\nu_{L,R})^C$. It is important to be careful with this notation since, $(\nu_{L,R})^C = (\nu^C)_{R,L}$, see page 70 in [21]. By using $(\overline{\psi}\phi)^\dagger = \phi^\dagger\overline{\psi}^\dagger = \phi^\dagger(\psi^\dagger\gamma^0)^\dagger = \phi^\dagger\gamma^{0\dagger}\psi = \phi^\dagger\gamma^0\psi = \overline{\phi}\psi$, we observe that $\left(\overline{\nu_L^C}\nu_L\right)^\dagger = \overline{\nu_L}\nu_L^C$ and the same holds for $L \leftrightarrow R$. We can then write down the Majorana mass term as,

$$\begin{aligned}\mathcal{L}_M &= -\frac{1}{2}m_L(\overline{\nu_L^C}\nu_L + \overline{\nu_L}\nu_L^C) - \frac{1}{2}m_R(\overline{\nu_R^C}\nu_R + \overline{\nu_R}\nu_R^C) \\ &= -\frac{1}{2}m_L\overline{\nu_{M,L}}\nu_{M,L} - \frac{1}{2}m_R\overline{\nu_{M,R}}\nu_{M,R}, \end{aligned} \tag{2.39}$$

where we have introduced the Majorana fields $\nu_{M,L} = \nu_L + \nu_L^C$ and $\nu_{M,R} = \nu_R + \nu_R^C$.

Dirac and Majorana Mass Term

Next, we can write down the most general Lagrangian mass term

$$\begin{aligned}\mathcal{L}_{D+M} = \mathcal{L}_D + \mathcal{L}_M &= -m_D(\overline{\nu_L}\nu_R + \overline{\nu_R}\nu_L) - \frac{1}{2}m_L(\overline{\nu_L^C}\nu_L + \overline{\nu_L}\nu_L^C) \\ &\quad -\frac{1}{2}m_R(\overline{\nu_R^C}\nu_R + \overline{\nu_R}\nu_R^C). \end{aligned} \tag{2.40}$$

In a $\mathcal{CP}$ invariant theory, all the mass parameters m_D, m_L, and m_R can be chosen to be real. By introducing the following notations

$$f_L = \frac{\nu_L + \nu_L^C}{\sqrt{2}}, \quad f_R = \frac{\nu_R + \nu_R^C}{\sqrt{2}} \tag{2.41}$$

the Lagrangian $\mathcal{L}_{D+M}$ can be written in a more compact form,

$$\mathcal{L}_{D+M} = \overline{V}MV, \tag{2.42}$$

where

$$V = \begin{pmatrix} f_L \\ f_R \end{pmatrix}, \quad M = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}. \tag{2.43}$$

Diagonalizing M gives the eigenvalues

$$m_{1,2} = \frac{m_L + m_R}{2} \pm \frac{1}{2}\sqrt{4m_D^2 + (m_L - m_R)^2}. \tag{2.44}$$

In a spontaneously broken gauge theory, couplings of the fermions to Higgs fields generate Dirac and Majorana mass terms.

The See-Saw Mechanism

The see-saw mechanism is used in GUTs, such as SO(10) and left-right symmetric models. In such models ν_R, acquires a large Majorana mass as a consequence of the symmetry breaking.

The left-right symmetric model, which is based on the gauge group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_Y$, contains a ν_R, this ν_R could be used to construct a Dirac and Majorana mass term. Spontaneous symmetry breaking in two steps.

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_Y \xrightarrow{\langle\Phi\rangle_R} \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \tag{2.45}$$

and the field W_R acquires a mass $M_{W_R} \sim \langle\Phi\rangle_R$.

$$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \xrightarrow{\langle\Phi\rangle_L} \mathrm{U}(1)_{\mathrm{EM}} \tag{2.46}$$

and the field W_L acquires a mass $M_{W_L} \sim \langle\Phi\rangle_L \sim 83$ GeV.
It is expected that $\langle\Phi\rangle_R \gg \langle\Phi\rangle_L$.

From the Dirac and Majorana mass term we have that the eigenvalues of the matrix M are $m_{1,2}$. Let us discuss the case when $m_L = 0$ and $m_R \gg m_D$. We then obtain the see-saw formulas

$$m_1 \simeq -\frac{m_D^2}{m_R}, \tag{2.47}$$

$$m_2 \simeq m_R. \tag{2.48}$$

In the case of more then one generation these results can be generalized so that m_D and m_R are matrices of the same dimension as the number of generations. To get an idea about the different sizes we consider the case when

$$m_D \simeq 250 \text{ GeV (electroweak scale)} \tag{2.49}$$

$$m_R \simeq 10^{16} \text{ GeV (GUT scale)}. \tag{2.50}$$

This gives that $m_1 \sim -6 \cdot 10^{-3}$ eV and $m_2 \sim 10^{16}$ GeV. The size of $|m_1|$ is of the right order of magnitude for neutrino oscillations.

Other Mass Models

Some other GUTs are based on the groups SU(5), SO(10), and E_6.

Since SO(10) contains the left-right symmetric gauge group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(4)_C$, it automatically contains right-handed neutrinos. Thus, in the SO(10) model, massive neutrinos arise naturally.

Supersymmetry (SUSY), i.e., the symmetry between bosons and fermions has been the subject of intense discussions in particle physics. In a supersymmetric theory, all known particles have a superpartner. If we assume that the theory of particles and forces must incorporate supersymmetry, this will have important consequences for the physics of neutrinos. Some supersymmetric models are the supersymmetric SM, the minimal supersymmetric SM (MSSM), and SUSY left-right models. These models are discussed in detail in [20].

2.5 Neutrino Experiments

Some experiments involving neutrino oscillations and neutrino decay are looking for processes which are forbidden if the neutrinos are massless while others as nuclear β-decay, pion decay and tau decay are based on processes which are allowed even in the SM with $m_\nu = 0$. We can thus divide the experiments into two different groups, which we will call kinematical tests and exclusive tests. We will here only discuss exclusive tests as neutrino oscillations.

2.5.1 Exclusive Tests

This group can be divided further into two subgroups; those experiments which depend on neutrino mixing and those in which mixing does not play a crucial role.

Neutrino Mixing Experiments

In neutrino oscillation experiments, one looks for effects when $x \neq L_{jk} \times m, m \in \mathbb{N}$, where L_{jk} are the oscillation lengths. There can be two types of experiments:

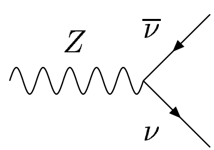
- Disappearance experiments: $P_{\alpha\beta}(x) < 1$, i.e., some of the ν_α of the original beam have disappeared.
- Appearance experiments: $P_{\alpha\beta}(x) > 0$ for $\alpha \neq \beta$, i.e., some of the ν_α of the original beam have disappeared which is compensated by the appearance of other flavors.

Other Experiments

The neutrinoless double β-decay process $n + n \to p + p + e^- + e^-$ violates the lepton number by 2 units. Thus, such a process is only possible with Majorana neutrinos.

2.5.2 How many Neutrinos?

In order to determine the number N_ν of light active left handed neutrinos one has to use the family structure of the SM and the assumption of a universal diagonal neutral-current coupling according to the figure below.



A model for N_ν is $N_\nu = \frac{\Gamma_{\text{inv}}}{\Gamma_\nu} = \frac{\Gamma_Z - \Gamma_h - 3\Gamma_l}{\Gamma_\nu}$, where Γ_{inv} is the decay rate of invisible neutrinos. Experiments at the Large Electron Positron collider (LEP) facility gives $N_\nu = 2.994 \pm 0.012$, see [1]. This clearly shows the existence of three neutrino flavors.

Chapter 3

Neutrinos Propagating through Vacuum

3.1 Basic Quantum Field Theory

Next, a short introduction of QFT will follow. For further details and many interesting questions see the excellent book by Peskin and Schroeder [24].

3.1.1 Notation and Basic Mathematical Tools

We will work with the units $\hbar = c = 1$. For the metric tensor we will use the convention of Bjorken and Drell [4],

$$g = (g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.1}$$

Four-vectors are written as $x^\mu = (x^0, \mathbf{x})$ and $x_\mu = g_{\mu\nu}x^\nu = (x^0, -\mathbf{x})$. The derivative operator is $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla\right)$ and we will use the notation $\partial_\mu \phi \equiv \phi_{,\mu}$. We will also define $\Box \equiv \partial^\mu \partial_\mu = \left(\frac{\partial}{\partial x^0}\right)^2 - \nabla^2$. Since the combination $\gamma \cdot A$ occurs so often we are going to use the notation $\not{A} \equiv \gamma^\mu A_\mu$ for any four-vector A_μ. We then have the useful relation

$$\begin{aligned} \not{A}\not{A} &= \gamma^\mu A_\mu \gamma^\nu A_\nu = (1/2)(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)A_\mu A_\nu \\ &= g^{\mu\nu}A_\mu A_\nu = A^2. \end{aligned} \tag{3.2}$$

The Heaviside step-function $\Theta(x)$ and the Dirac delta-function $\delta(x)$ are defined as follows

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}, \tag{3.3}$$

$$\int d^n x \delta^{(n)}(\mathbf{x}) = 1. \tag{3.4}$$

The following identity is important: $\int d^4 x e^{ikx} = (2\pi)^4 \delta^{(4)}(k)$.
For the anticommutator we will use the notation $[A, B]_+ = AB + BA$.
The Pauli sigma matrices are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.5}$$

and we will define $\boldsymbol{\sigma} \equiv (\sigma^1, \sigma^2, \sigma^3)$.

3.1.2 Lagrangian Formulation of Field Theory

In classical mechanics, the fundamental quantity is the action S:

$$S = \int L dt = \int \mathcal{L}(\phi_r, \partial_\mu \phi_r) d^4 x. \tag{3.6}$$

Here S has been written as an integral over a Lagrangian density $\mathcal{L}$. This is possible in a local field theory. That the Lagrangian density should depend only on one or more fields $\phi_r(x)$ and their derivatives $\partial_\mu \phi_r(x)$, where r labels the different fields, is not the most general case, but it greatly simplifies the formalism.

The principle of least action, $\delta S = 0$, gives the Euler-Lagrange equation of motion for a field,

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\mu}} \right) = 0. \tag{3.7}$$

In relativistic dynamics, the Lagrangian formulation of field theory is particularly good, since all expressions are explicitly Lorentz invariant.

Next, let $\pi(x)$ be the momentum density conjugate to $\phi(x)$ defined according to $\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$. Then, the Hamiltonian can be written as

$$H = \int \left[\pi(x) \dot{\phi}(x) - \mathcal{L} \right] d^3 \mathbf{x} \equiv \int \mathcal{H} d^3 \mathbf{x}. \tag{3.8}$$

3.1.3 The Klein-Gordon Field

Let us next consider the Lagrangian density $\mathcal{L}(x) = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$ for a single real scalar field $\phi(x)$. The Euler-Lagrange equation then gives the equation of motion for this field, $\left(\Box + m^2\right) \phi(x) = 0$. This is the Klein-Gordon equation.
The field $\phi(x)$ can be used to describe massive particles with spin 0, for example π-mesons. If one wants to describe charged massive particles with spin 0, one has to use a complex scalar field.

3.1.4 The Dirac Field

Consider the Lagrangian density $\mathcal{L}(x) = \overline{\psi}(x)(i\partial\!\!\!/ - m)\psi(x)$, where $\partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$ and the adjoint field is $\overline{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$. The Dirac equation $(i\partial\!\!\!/ - m)\psi(x) = 0$ is derived by using the Euler-Lagrange equation on the adjoint field $\overline{\psi}(x)$.

The Dirac γ-matrices satisfy the anticommutation relations,

$$[\gamma^\mu, \gamma^\nu]_+ \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1} \qquad (3.9)$$

and the condition $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$. There are several 4×4 matrix representations of the γ-matrices in four-dimensional Minkowski space, for example the Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \qquad (3.10)$$

and the chiral representation (also called the Weyl representation)

$$\gamma^0 = \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \qquad (3.11)$$

where $\mathbb{1}$ denotes the 2×2 unit matrix and σ^j denotes the Pauli matrices.
All the 4×4 representations of the Dirac algebra are unitarily equivalent, i.e., knowing one representation of the gamma matrices is enough, since all other ones can be obtained in the following way $\gamma'_\mu = U\gamma_\mu U^\dagger$, where U is a unitary matrix. For example, the Majorana representation is obtained by using

$$U = U^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \sigma^2 \\ \sigma^2 & -\mathbb{1} \end{pmatrix}. \qquad (3.12)$$

We also introduce the matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$, which has the following properties

$$[\gamma^\mu, \gamma^5]_+ = 0, \quad (\gamma^5)^2 = \mathbb{1}, \quad \gamma^{5\dagger} = \gamma^5, \quad \gamma_5 = \gamma^5. \qquad (3.13)$$

Here we have used the totally antisymmetric Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$. There are of course many other nice relations involving contractions and traces of γ-matrices, but we will not present them here.

The Dirac equation describes a massive particle with spin 1/2, for example, an electron. Suppose that $\psi(x)$ satisfies the Dirac equation. Then, it is easy to show that it also satisfies the Klein-Gordon equation by acting with $(-i\partial\!\!\!/ - m)$ on the Dirac equation. Since

$$(-i\partial\!\!\!/ - m)(i\partial\!\!\!/ - m)\psi(x) = (\partial\!\!\!/\partial\!\!\!/ + m^2)\psi(x) = (\partial^2 + m^2)\psi(x), \qquad (3.14)$$

we have

$$(\partial^2 + m^2)\psi(x) = 0, \qquad (3.15)$$

which proves that $\psi(x)$ satisfies the Klein-Gordon equation. Since a Dirac field $\psi(x)$ obeys the Klein-Gordon equation, it can be written as a linear combination of plane waves, i.e.,

$$\psi(x) = u(p)e^{-ipx}, \quad p^2 = m^2, \quad p^0 > 0. \tag{3.16}$$

Inserting this into the Dirac equation gives $(\not{p} - m)u(p) = 0$. There are two linearly independent solutions for $u(p)$, which we will denote as $u^s(p)$. These obey the following normalization conditions: $u^{r\dagger}(p)u^s(p) = 2E(\mathbf{p})\delta^{rs}$. Note that $\psi(x) = u^s(p)e^{-ipx}$ are called positive frequency solutions. In the same, way one can find the negative frequency solutions

$$\psi(x) = v^s(p)e^{ipx}, \quad p^2 = m^2, \quad p^0 > 0. \tag{3.17}$$

The results of the quantized Dirac theory can be summarized as follows

$$\nu_j(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_j(\mathbf{p})}} \sum_s \left(a_j^s(\mathbf{p}) u_j^s(p) e^{-ipx} + b_j^{s\dagger}(\mathbf{p}) v_j^s(p) e^{ipx} \right) \tag{3.18}$$

and

$$\overline{\nu}_j(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_j(\mathbf{p})}} \sum_s \left(b_j^s(\mathbf{p}) \overline{v}_j^s(p) e^{-ipx} + a_j^{s\dagger}(\mathbf{p}) \overline{u}_j^s(p) e^{ipx} \right), \tag{3.19}$$

where

$$\left[a_j^r(\mathbf{p}), a_j^{s\dagger}(\mathbf{q})\right]_+ = \left[b_j^r(\mathbf{p}), b_j^{s\dagger}(\mathbf{q})\right]_+ = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})\delta^{rs}, \tag{3.20}$$

$$\text{all other } [\cdots, \cdots]_+ = 0, \tag{3.21}$$

$$u_j^{r\dagger}(p)u_j^s(p) = 2E_j(\mathbf{p})\delta^{rs}, \quad v_j^{r\dagger}(p)v_j^s(p) = 2E_j(\mathbf{p})\delta^{rs}, \tag{3.22}$$

$$\overline{u}_j^r(p)u_j^s(p) = 2m_j\delta^{rs}, \quad \overline{v}_j^r(p)v_j^s(p) = -2m_j\delta^{rs}, \tag{3.23}$$

$$\overline{u}_j^r(p)v_j^s(p) = \overline{v}_j^r(p)u_j^s(p) = 0, \tag{3.24}$$

$$u_j^{r\dagger}(p^0, \mathbf{p})v_j^s(p^0, -\mathbf{p}) = v_j^{r\dagger}(p^0, -\mathbf{p})u_j^s(p^0, \mathbf{p}) = 0, \tag{3.25}$$

$$(\not{p} - m_j)u_j^s(p) = 0, \quad (\not{p} + m_j)v_j^s(p) = 0, \tag{3.26}$$

$$\overline{u}_j^r(p) = u_j^{r\dagger}(p)\gamma^0, \quad \overline{v}_j^r(p) = v_j^{r\dagger}(p)\gamma^0, \tag{3.27}$$

$$\not{p} + m_j = \sum_s u_j^s(p)\overline{u}_j^s(p), \quad \not{p} - m_j = \sum_s v_j^s(p)\overline{v}_j^s(p), \tag{3.28}$$

$$|\nu_j^r(\mathbf{p})\rangle = \sqrt{2E_j(\mathbf{p})}\, a_j^{r\dagger}(\mathbf{p})|0\rangle, \tag{3.29}$$

$$\langle \nu_j^s(\mathbf{q})|\nu_j^r(\mathbf{p})\rangle = 2E_j(\mathbf{p})(2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q})\delta^{rs}. \tag{3.30}$$

The indices s and r denote the spin $s, r = \pm 1/2$. Here $a_j^s(\mathbf{p})$ is the annihilation operator for a particle with spin s and momentum $\mathbf{p}$, similarly, $b_j^{s\dagger}(\mathbf{p})$ is the creation operator for an antiparticle with spin s and momentum $\mathbf{p}$. Note that we have put an index j everywhere, this is done in order to be able to distinguish different neutrino mass fields later on.

3.2 Quantum Field Theoretical Treatment of Neutrino Oscillations

3.2.1 Lagrangian Density

From the complete Lagrangian density of the standard electroweak theory in the unitary gauge, the following part is of interest to us

$$\begin{aligned}\mathcal{L}(x) &= \sum_{\beta=e,\mu,\tau} \overline{\psi}_\beta(x)(i\partial\!\!\!/ - m_\beta)\psi_\beta(x) \\ &\quad + \sum_{j=1}^{3} \overline{\nu}_j(x)(i\partial\!\!\!/ - m_j)\nu_j(x) + \mathcal{L}_{\text{int}}(x). \end{aligned} \tag{3.31}$$

The first two terms correspond to the usual kinetic part, while the interaction part is given by

$$\begin{aligned}\mathcal{L}_{\text{int}}(x) &= \sum_{\beta=e,\mu,\tau} \Big[-\frac{g}{2\sqrt{2}} \overline{\psi}_{\nu_\beta}(x)\gamma^\alpha(\mathbb{1}-\gamma^5)\psi_\beta(x)W_\alpha(x) \\ &\quad - \frac{g}{2\sqrt{2}} \overline{\psi}_\beta(x)\gamma^\alpha(\mathbb{1}-\gamma^5)\psi_{\nu_\beta}(x)W^\dagger_\alpha(x) \\ &\quad - \frac{g}{4\cos\theta_W} \overline{\psi}_{\nu_\beta}(x)\gamma^\alpha(\mathbb{1}-\gamma^5)\psi_{\nu_\beta}(x)Z_\alpha(x) \\ &\quad - \frac{1}{V} m_{\nu_\beta}\overline{\psi}_{\nu_\beta}(x)\psi_{\nu_\beta}(x)\sigma \Big] \end{aligned} \tag{3.32}$$

where g is a dimensionless coupling constant and $W_\alpha(x)$ is the field that describes the W particles, which are vector bosons with spin 1. The simplest equation for a vector field $W_\alpha(x)$ describing particles of mass m_W and spin 1 is the Proca equation. We observe that the interaction part contains the flavor eigenfields ψ_{ν_β}. These fields are related to the mass eigenfields ν_j by the relation $\psi_{\nu_\beta} = \sum_j U_{\beta j}\nu_j$. Since $\overline{\psi}(x) = \psi^\dagger(x)\gamma^0$, $\overline{\psi}_{\nu_\beta}$ can be written as

$$\overline{\psi}_{\nu_\beta} = \left(\sum_j U_{\beta j}\nu_j \right)^\dagger \gamma^0 = \sum_j U^*_{\beta j}\nu^\dagger_j\gamma^0 = \sum_j U^*_{\beta j}\overline{\nu}_j. \tag{3.33}$$

Inserting the expressions for ψ_{ν_β} and $\overline{\psi}_{\nu_\beta}$ expressed in terms of the mass eigenfields into the interaction Lagrangian, one obtains

$$\begin{aligned}\mathcal{L}_{\text{int}}(x) &= \sum_{\beta=e,\mu,\tau}\sum_{j=1}^{3} \Big[-\frac{g}{2\sqrt{2}} \left[U^*_{\beta j}\overline{\nu}_j(x)\gamma^\alpha(\mathbb{1}-\gamma^5)\psi_\beta(x)W_\alpha(x) \right. \\ &\quad \left. + \overline{\psi}_\beta(x)\gamma^\alpha(\mathbb{1}-\gamma^5)U_{\beta j}\nu_j(x)W^\dagger_\alpha(x) \right] \\ &\quad - \frac{g}{4\cos\theta_W} \sum_{k=1}^{3} U^*_{\beta j}\overline{\nu}_j(x)\gamma^\alpha(\mathbb{1}-\gamma^5)U_{\beta k}\nu_k(x)Z_\alpha(x) \Big]. \end{aligned} \tag{3.34}$$

We have skipped the part of the interaction Lagrangian containing the σ field.

To obtain the equation of motion for the field ν_j, we have to use the Euler-Lagrange equation and take the derivatives with respect to the field $\overline{\nu}_j$. Since we observe that $\mathcal{L}$ does not depend on $\overline{\nu}_{j,\mu}$, we obtain

$$\frac{\partial \mathcal{L}}{\partial \overline{\nu}_{j,\mu}} = 0 \Rightarrow \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \overline{\nu}_{j,\mu}}\right) = 0. \tag{3.35}$$

Next, we take the derivative of $\mathcal{L}$ with respect to $\overline{\nu}_j$. This gives

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \overline{\nu}_j} &= (i\partial\!\!\!/ - m_j)\nu_j(x) - \frac{g}{2\sqrt{2}}\sum_{\beta=e,\mu,\tau} U^*_{\beta j}\gamma^\alpha(\mathbb{1}-\gamma^5)\psi_\beta(x)W_\alpha(x)\\ &\quad - \frac{g}{4\cos\theta_W}\sum_{\beta=e,\mu,\tau}\sum_{k=1}^{3} U^*_{\beta j}\gamma^\alpha(\mathbb{1}-\gamma^5)U_{\beta k}\nu_k(x)Z_\alpha(x).\end{aligned} \tag{3.36}$$

The Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \overline{\nu}_j} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \overline{\nu}_{j,\mu}}\right) = 0 \tag{3.37}$$

gives the equation of motion for ν_j, which can be written in the following form

$$(i\partial\!\!\!/ - m_j)\nu_j(x) = \chi_j(x), \tag{3.38}$$

where we have introduced the source term $\chi_j(x)$

$$\begin{aligned}\chi_j(x) &= \sum_{\beta=e,\mu,\tau}\left[\frac{g}{2\sqrt{2}}U^*_{\beta j}\gamma^\alpha(\mathbb{1}-\gamma^5)\psi_\beta(x)W_\alpha(x)\right.\\ &\quad \left. + \frac{g}{4\cos\theta_W}\sum_{k=1}^{3} U^*_{\beta j}\gamma^\alpha(\mathbb{1}-\gamma^5)U_{\beta k}\nu_k(x)Z_\alpha(x)\right].\end{aligned} \tag{3.39}$$

Thus, we see that the different neutrino fields $\nu_j(x)$ are coupled to each other.

3.2.2 The Flavor Eigenstates in a Wave Packet Approach

We would like to solve the equation of motion for $\nu_j(x)$, i.e., to obtain an explicit expression for $\nu_j(x)$. This is unfortunately not possible. The best thing we can do is to use perturbation theory and expand $\nu_j(x)$ in the coupling constant g. To zeroth order in the coupling constant g, i.e. setting $g = 0$ in Eq. (3.39), we obtain uncoupled Dirac equations $(i\partial\!\!\!/ - m_j)\nu_j(x) = 0$, which have the following solutions

$$\nu_j(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_j(\mathbf{p})}}\sum_s \left(a^s_j(\mathbf{p})u^s_j(p)e^{-ipx} + b^{s\dagger}_j(\mathbf{p})v^s_j(p)e^{ipx}\right). \tag{3.40}$$

Here we have only written down the field $\nu_j(x)$, since this and the conjugate field $\overline{\nu}_j(x)$ can be found in section 3.1.4 treating the Dirac field. Thus, we have expanded the neutrino mass field in terms of the operators $a^s_j(\mathbf{p})$ and $b^{s\dagger}_j(\mathbf{p})$.

However, we are interested in having an expression for the creation operator for a neutrino with definite momentum $\mathbf{p}$ and spin s as an expression involving the fields $\nu_j(x)$ and $\overline{\nu}_j(x)$. Using appendix A, we find that the annihilation operator for a neutrino, $a_j^r(\mathbf{q})$, can be expressed in terms of the field $\nu_j(x)$ according to

$$a_j^r(\mathbf{q}) = \frac{1}{\sqrt{2E_j(\mathbf{q})}} u_j^{r\dagger}(q) \int d^3\mathbf{x} e^{iqx} \nu_j(x). \tag{3.41}$$

Thus, the corresponding creation operator is obtained by taking the Hermitian conjugate of this, i.e.,

$$a_j^{r\dagger}(\mathbf{q}) = \frac{1}{\sqrt{2E_j(\mathbf{q})}} u_j^{r}(q) \int d^3\mathbf{x} e^{-iqx} \nu_j^\dagger(x). \tag{3.42}$$

A neutrino state $|\nu_j^r(\mathbf{p})\rangle$ with mass m_j, momentum $\mathbf{p}$ and spin r is obtained by acting with $a_j^{r\dagger}(\mathbf{p})$ on the vacuum state $|0\rangle$. The vacuum state $|0\rangle$ is defined such that $\langle 0|0\rangle = 1$. In order for $|\nu_j^r(\mathbf{p})\rangle$ to be properly normalized, i.e., $\langle \nu_j^s(\mathbf{q})|\nu_j^r(\mathbf{p})\rangle = 2E_j(\mathbf{p})(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}$, the definition for $|\nu_j^r(\mathbf{p})\rangle$ is

$$|\nu_j^r(\mathbf{p})\rangle \equiv \nu_j^{r\dagger}(\mathbf{p})|0\rangle \equiv \sqrt{2E_j(\mathbf{p})} a_j^{r\dagger}(\mathbf{p})|0\rangle, \tag{3.43}$$

where $E_j(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_j^2}$. The flavor state basis and the mass eigenstate basis are denoted by $\mathcal{H}_{\text{flavor}} \equiv \{|\psi_\alpha\rangle\}_{\alpha=e,\mu,\tau}$ and $\mathcal{H}_{\text{mass}} \equiv \{|\nu_j\rangle\}_{j=1,2,3}$, respectively. The neutrino flavor fields ψ_α^r, where $\alpha = e,\mu,\tau$, are expressed in the mass eigenfields ν_j^r, where $j = 1,2,3$, by $\psi_\alpha^r(\mathbf{p}) = \sum_j U_{\alpha j}\nu_j^r(\mathbf{p})$. The $U_{\alpha j}$'s are the elements of a unitary mixing matrix U, $U^\dagger U = UU^\dagger = \mathbb{1}$. Taking the Hermitian conjugate of $\psi_\alpha^r(\mathbf{p}) = \sum_j U_{\alpha j}\nu_j^r(\mathbf{p})$ yields $\psi_\alpha^{r\dagger}(\mathbf{p}) = \sum_j U_{\alpha j}^*\nu_j^{r\dagger}(\mathbf{p})$. If we let this expression act on the vacuum state $|0\rangle$, find that $|\psi_\alpha^r(\mathbf{p})\rangle = \sum_j U_{\alpha j}^*|\nu_j^r(\mathbf{p})\rangle$. Since we want to have an expression for $|\psi_\alpha^r(\mathbf{p})\rangle$ in terms of $\nu_j^{r\dagger}$, we obtain

$$|\psi_\alpha^r(\mathbf{p})\rangle = \sum_j U_{\alpha j}^* \sqrt{2E_j(\mathbf{p})} a_j^{r\dagger}(\mathbf{p})|0\rangle = \sum_j U_{\alpha j}^* u_j^r(p) \int d^3\mathbf{x}' e^{-ipx'} \nu_j^{r\dagger}(x')|0\rangle. \tag{3.44}$$

The state $|\psi_\alpha^r(\mathbf{p})\rangle$ has a definite momentum $\mathbf{p}$, which means that $\Delta\mathbf{p} = \mathbf{0}$. According to Heisenberg's uncertainty relation, $\Delta p \Delta x = \hbar/2$, the state has therefore an infinitely wide spread in space, i.e., $\Delta\mathbf{x} = (\infty,\infty,\infty)$. In order to have a state that is localized around $\mathbf{X},\mathbf{P},T$, one has to use a wave-packet with some distribution function $F(\mathbf{X},\mathbf{P},T,\mathbf{p})$. We can then write a state $|\psi_\alpha^r(\mathbf{X},\mathbf{P},T)\rangle$ that is localized around $\mathbf{X},\mathbf{P},T$ as

$$|\psi_\alpha^r(\mathbf{X},\mathbf{P},T)\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\alpha(\mathbf{p})}} F(\mathbf{X},\mathbf{P},T,\mathbf{p})|\psi_\alpha^r(\mathbf{p})\rangle. \tag{3.45}$$

Inserting our expression for $|\psi_\alpha^r(\mathbf{p})\rangle$ we finally have

$$|\psi_\alpha^r(\mathbf{X},\mathbf{P},T)\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{F(\mathbf{X},\mathbf{P},T,\mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}}\sum_j U^*_{\alpha j}u_j^r(p)\int d^3\mathbf{x}' e^{-ipx'}\nu_j^{r\dagger}(x')|0\rangle. \tag{3.46}$$

3.2.3 The Amplitude

The transition amplitude $A^{rs}_{\alpha\beta}(\mathbf{X},\mathbf{Y},T_p,T_d)$ for a neutrino of flavor α to oscillate into flavor β is given by

$$A^{rs}_{\alpha\beta}(\ldots) = \langle\psi_\beta^s(\mathbf{Y},\mathbf{Q},T_d)|\psi_\alpha^r(\mathbf{X},\mathbf{P},T_p)\rangle. \tag{3.47}$$

Using the expressions for $|\psi_\alpha^r(\mathbf{X},\mathbf{P},T_p)\rangle$ and $\langle\psi_\beta^s(\mathbf{Y},\mathbf{Q},T_d)| = |\psi_\beta^s(\mathbf{Y},\mathbf{Q},T_d)\rangle^\dagger$, we have

$$\begin{aligned}
A^{rs}_{\alpha\beta}(\cdots) &= \langle 0|\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{F^\dagger(\mathbf{Y},\mathbf{Q},T_d,\mathbf{q})}{\sqrt{2E_\beta(\mathbf{q})}}\sum_j U_{\beta j}u_j^{s\dagger}(q)\int d^3\mathbf{y}' e^{iqy'}\nu_j^s(y') \\
&\quad \times\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{F(\mathbf{X},\mathbf{P},T_p,\mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}}\sum_k U^*_{\alpha k}u_k^r(p)\int d^3\mathbf{x}' e^{-ipx'}\nu_k^{r\dagger}(x')|0\rangle \\
&= \int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{F^\dagger(\mathbf{Y},\mathbf{Q},T_d,\mathbf{q})}{\sqrt{2E_\beta(\mathbf{q})}}\sum_j U_{\beta j}u_j^{s\dagger}(q)\int d^3\mathbf{y}' e^{iqy'} \\
&\quad \times\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{F(\mathbf{X},\mathbf{P},T_p,\mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}}\sum_k U^*_{\alpha k}u_k^r(p)\int d^3\mathbf{x}' e^{-ipx'} \\
&\quad \times\langle 0|\nu_j^s(y')\nu_k^{r\dagger}(x')|0\rangle
\end{aligned} \tag{3.48}$$

The expression $\langle 0|\nu_j^s(y')\nu_k^{r\dagger}(x')|0\rangle$ can be interpreted as a mass eigenstate k being created at x', propagating to y' where a mass eigenstate j is being annihilated. Note that j has to be equal to k, since there is no way that the mass eigenstate k can propagate into a different mass eigenstate j, at least not in vacuum. On the other hand, when neutrinos propagate through matter we cannot simply assume that $j = k$. Since $y'^0 > x'^0$, we can insert the time-ordering operator T defined by

$$T(\phi(y')\phi(x')) = \begin{cases} \phi(y')\phi(x'), & \text{if } y'^0 > x'^0 \\ -\phi(x')\phi(y'), & \text{if } x'^0 > y'^0 \end{cases}. \tag{3.49}$$

We thus obtain

$$\langle 0|\nu_j^s(y')\nu_k^{r\dagger}(x')|0\rangle \to \langle 0|T(\nu_j^s(y')\nu_j^{r\dagger}(x'))|0\rangle. \tag{3.50}$$

We can rewrite the last expression in the following way

$$\begin{aligned}\langle 0|T(\nu_j^s(y')\nu_j^{r\dagger}(x'))|0\rangle &= \langle 0|T(\nu_j^s(y')\nu_j^{r\dagger}(x')\underbrace{\gamma^0\gamma^0}_{=\mathbb{1}})|0\rangle = \{\overline{\nu} = \nu^\dagger\gamma^0\} \\ &= \langle 0|T(\nu_j^s(y')\overline{\nu}_j^{r\dagger}(x'))|0\rangle\gamma^0 = S_j(y'-x')\gamma^0,\end{aligned} \tag{3.51}$$

where we have introduced the Feynman fermion propagator defined by

$$S_j(y'-x') = \int \frac{d^4k}{(2\pi)^4}\frac{i(\not{k}+m_j)}{k^2-m_j^2+i\epsilon}e^{-ik(y'-x')}. \tag{3.52}$$

The integration in the complex k^0 plane has to be taken along the whole real axis $-\infty < k^0 < \infty$. We can now rewrite the expression for the amplitude as follows

$$\begin{aligned}A_{\alpha\beta}^{rs}(\cdots) &= \sum_j U_{\beta j}U_{\alpha j}^* \int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{F^\dagger(\mathbf{Y},\mathbf{Q},T_d,\mathbf{q})}{\sqrt{2E_\beta(\mathbf{q})}} \\ &\times \int d^3\mathbf{y}' e^{iqy'} u_j^{s\dagger}(q) \\ &\times \int \frac{d^4k}{(2\pi)^4}\frac{i(\not{k}+m_j)}{k^2-m_j^2+i\epsilon}e^{-ik(y'-x')}\gamma^0 \\ &\times \int d^3\mathbf{x}' e^{-ipx'} u_j^r(p) \\ &\times \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{F(\mathbf{X},\mathbf{P},T_p,\mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}}.\end{aligned} \tag{3.53}$$

The question is now in which order we should calculate the integrals. We observe that the spinors $u_j^{s\dagger}(q)$ and $u_j^r(p)$ can be moved out from the three integrals in the middle. Thus, we can use the result from appendix B which says

$$\begin{aligned}&\int d^3\mathbf{y}' e^{iqy'} \int \frac{d^4k}{(2\pi)^4}\frac{i(\not{k}+m_j)}{k^2-m_j^2+i\epsilon}e^{-ik(y'-x')} \int d^3\mathbf{x}' e^{-ipx'} \\ &= -(2\pi)^2\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{-ix'^0p^0}e^{iy'^0q^0}\,(\not{p}+m_j)\,\frac{\pi e^{-i(y'^0-x'^0)E_j(\mathbf{p})}}{E_j(\mathbf{p})}.\end{aligned} \tag{3.54}$$

It follows that

$$\begin{aligned}A_{\alpha\beta}^{rs}(\cdots) &= -\sum_j U_{\beta j}U_{\alpha j}^* \int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{F^\dagger(\mathbf{Y},\mathbf{Q},T_d,\mathbf{q})}{\sqrt{2E_\beta(\mathbf{q})}} \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{F(\mathbf{X},\mathbf{P},T_p,\mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}} \\ &\times(2\pi)^2\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{-ix'^0p^0}e^{iy'^0q^0} \\ &\times u_j^{s\dagger}(q)\,(\not{p}+m_j)\,\frac{\pi e^{-i(y'^0-x'^0)E_j(\mathbf{p})}}{E_j(\mathbf{p})}\gamma^0 u_j^r(p).\end{aligned} \tag{3.55}$$

Since $p_0 = E_k(\mathbf{p})$, $q_0 = E_j(\mathbf{q})$ and as stated earlier $j = k$, we have that $p_0 = E_j(\mathbf{p})$ and $q_0 = E_j(\mathbf{q})$. Thus we have

$$\begin{aligned}
A^{rs}_{\alpha\beta}(\cdots) &= -\sum_j U_{\beta j}U^*_{\alpha j}\frac{\pi}{(2\pi)^4}\int d^3\mathbf{q}d^3\mathbf{p}\frac{F^\dagger(\mathbf{Y},\mathbf{Q},T_d,\mathbf{q})}{\sqrt{2E_\beta(\mathbf{q})}}\frac{F(\mathbf{X},\mathbf{P},T_p,\mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}} \\
&\quad\times\delta^{(3)}(\mathbf{p}-\mathbf{q})\frac{e^{-iy'^0(E_j(\mathbf{p})-E_j(\mathbf{q}))}}{E_j(\mathbf{p})}u^{s\dagger}_j(q)\left(\not p+m_j\right)\gamma^0 u^r_j(p) \\
&= -\sum_j U_{\beta j}U^*_{\alpha j}\frac{\pi}{(2\pi)^4}\int d^3\mathbf{p}\frac{F^\dagger(\mathbf{Y},\mathbf{Q},T_d,\mathbf{p})}{\sqrt{2E_\beta(\mathbf{p})}}\frac{F(\mathbf{X},\mathbf{P},T_p,\mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}} \\
&\quad\times\frac{u^{s\dagger}_j(p)\left(\not p+m_j\right)\gamma^0 u^r_j(p)}{E_j(\mathbf{p})}. \qquad (3.56)
\end{aligned}$$

The part $u^{s\dagger}_j(p)\left(\not p+m_j\right)\gamma^0 u^r_j(p)$ can be simplified by using that $\not p + m_j$ can be written as a sum of spinors according to

$$\not p + m_j = \sum_s u^s_j(p)\overline{u}^s_j(p) = \sum_s u^s_j(p)u^{s\dagger}_j(p)\gamma^0. \qquad (3.57)$$

Using this gives

$$\begin{aligned}
&u^{s\dagger}_j(p)\left(\not p+m_j\right)\gamma^0 u^r_j(p) = u^{s\dagger}_j(p)\sum_t u^t_j(p)u^{t\dagger}_j(p)\gamma^0\gamma^0 u^r_j(p) \\
&= \{(\gamma^0)^2 = 1 \quad \text{and} \quad u^{r\dagger}_j(p)u^s_j(p) = 2E_j(\mathbf{p})\delta^{rs}\} \\
&= u^{s\dagger}_j(p)\sum_t u^t_j(p)2E_j(\mathbf{p})\delta^{tr} = u^{s\dagger}_j(p)u^r_j(p)2E_j(\mathbf{p}) \\
&= (2E_j(\mathbf{p}))^2\delta^{rs}. \qquad (3.58)
\end{aligned}$$

We finally arrive at a very simple expression for the transition amplitude

$$A^{rs}_{\alpha\beta}(\cdots) = -\sum_j U_{\beta j}U^*_{\alpha j}\delta^{rs}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{F^\dagger(\mathbf{Y},\mathbf{Q},T_d,\mathbf{p})}{\sqrt{2E_\beta(\mathbf{p})}}\frac{F(\mathbf{X},\mathbf{P},T_p,\mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}}2E_j(\mathbf{p}). \qquad (3.59)$$

An important question is what we mean by $E_\alpha(\mathbf{p})$, i.e., in what way should we define the energy for a certain flavor α. From the condition that our states have to be normalized, i.e.,

$$\langle\psi^r_\alpha(\mathbf{X},\mathbf{P},T)|\psi^r_\alpha(\mathbf{X},\mathbf{P},T)\rangle = 1, \qquad (3.60)$$

we have from appendix D that $E_\alpha(\mathbf{p}) = \sum_j U_{\alpha j}U^*_{\alpha j}E_j(\mathbf{p})$.

3.2.4 Gaussian Wave Packet

In order to proceed further we need to have an expression for the distribution function $F(\mathbf{X}, \mathbf{P}, T, \mathbf{p})$. We will assume

$$F(\mathbf{X}, \mathbf{P}_j, T_p, \mathbf{p}) = \left(\frac{2\pi}{\sigma_p^2}\right)^{3/4} e^{-\frac{(\mathbf{p}-\mathbf{P}_j)^2}{4\sigma_p^2}} e^{-i(\mathbf{p}\cdot\mathbf{X}-E_j(\mathbf{p})T_p)}, \tag{3.61}$$

i.e., we have a Gaussian wave packet where the width of the wave packet σ_p is the same along all the three directions. Furthermore, assume that a neutrino with flavor α and spin r is produced at $(T_p, \mathbf{X})$ with a momentum uncertainty of σ_p around $\mathbf{P}$ and that a neutrino with flavor β and spin s is detected at $(T_d, \mathbf{Y})$ with a momentum uncertainty of σ_d around $\mathbf{Q}$. Then, the amplitude becomes

$$\begin{aligned} A_{\alpha\beta}^{rs}(\cdots) &= -\sum_j U_{\beta j}U_{\alpha j}^*\delta^{rs}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{2E_j(\mathbf{p})}{\sqrt{2E_\beta(\mathbf{p})}\sqrt{2E_\alpha(\mathbf{p})}} \\ &\quad\times\left(\frac{2\pi}{\sigma_p^2}\right)^{3/4}\left(\frac{2\pi}{\sigma_d^2}\right)^{3/4} e^{-\frac{(\mathbf{p}-\mathbf{P}_j)^2}{4\sigma_p^2}-\frac{(\mathbf{p}-\mathbf{Q}_j)^2}{4\sigma_d^2}} \\ &\quad\times e^{i\mathbf{p}\cdot(\mathbf{Y}-\mathbf{X})}e^{-iE_j(\mathbf{p})(T_d-T_p)}. \end{aligned} \tag{3.62}$$

The Amplitude in the Relativistic Limit

In the relativistic limit, $E_j(\mathbf{p}) = \sqrt{\mathbf{p}^2+m_j^2} \approx |\mathbf{p}| + \frac{m_j^2}{2|\mathbf{p}|}$, we make the following approximations

$$\begin{aligned} &\frac{2E_j(\mathbf{p})}{\sqrt{2E_\beta(\mathbf{p})}\sqrt{2E_\alpha(\mathbf{p})}} \\ &= \left\{E_j(\mathbf{p}) \approx |\mathbf{p}| \Rightarrow E_\alpha(\mathbf{p}) = \sum_j U_{\alpha j}^*U_{\alpha j}E_j(\mathbf{p}) \approx \sum_j U_{\alpha j}^*U_{\alpha j}|\mathbf{p}| = |\mathbf{p}|\right\} \\ &= \frac{2|\mathbf{p}|}{\sqrt{2|\mathbf{p}|}\sqrt{2|\mathbf{p}|}} = 1. \end{aligned} \tag{3.63}$$

Using this approximation in the expression for the amplitude, we obtain

$$\begin{aligned} A_{\alpha\beta}^{rs}(\cdots) &= -\sum_j U_{\beta j}U_{\alpha j}^*\delta^{rs}\left(\frac{2\pi}{\sigma_p^2}\right)^{3/4}\left(\frac{2\pi}{\sigma_d^2}\right)^{3/4} \\ &\quad\times\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{-\frac{(\mathbf{p}-\mathbf{P}_j)^2}{4\sigma_p^2}-\frac{(\mathbf{p}-\mathbf{Q}_j)^2}{4\sigma_d^2}}e^{i\mathbf{p}\cdot(\mathbf{Y}-\mathbf{X})}e^{-iE_j(\mathbf{p})(T_d-T_p)}. \end{aligned} \tag{3.64}$$

In order to simplify the calculations, we introduce the following notations

$$\mathbf{L} = \mathbf{Y} - \mathbf{X}, \quad T = T_d - T_p, \quad A = \frac{1}{4\sigma_p^2} + \frac{1}{4\sigma_d^2}, \quad \mathbf{B}_j = \frac{\mathbf{P}_j}{4\sigma_p^2} + \frac{\mathbf{Q}_j}{4\sigma_d^2},$$
$$C_j = \frac{\mathbf{P}_j^2}{4\sigma_p^2} + \frac{\mathbf{Q}_j^2}{4\sigma_d^2}. \tag{3.65}$$

Thus the amplitude can be written

$$\begin{aligned} A_{\alpha\beta}^{rs}(\cdots) &= -\sum_j U_{\beta j} U_{\alpha j}^* \delta^{rs} \left(\frac{2\pi}{\sigma_p^2}\right)^{3/4} \left(\frac{2\pi}{\sigma_d^2}\right)^{3/4} e^{\mathbf{B}_j^2/A - C_j} \\ &\times \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-A(\mathbf{p}-\mathbf{B}_j/A)^2} e^{i\mathbf{p}\cdot\mathbf{L}} e^{-iE_j(\mathbf{p})T}. \end{aligned} \tag{3.66}$$

This integral cannot be solved exactly due to the fact that $E_j(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_j^2}$. The only way around this problem is to use some approximation techniques. Since the Gaussian wave packet in momentum space is peaked around the average momentum $\langle \mathbf{p} \rangle_j = \mathbf{B}_j/A$, we can make the approximations

$$E_j(\mathbf{p}) \approx \langle E_j \rangle + \mathbf{v}_j \cdot (\mathbf{p} - \langle \mathbf{p} \rangle_j), \tag{3.67}$$
$$\langle E_j \rangle = E_j(\langle \mathbf{p} \rangle_j) = \sqrt{\langle \mathbf{p} \rangle_j^2 + m_j^2}, \tag{3.68}$$
$$\mathbf{v}_j = \frac{\langle \mathbf{p} \rangle_j}{\langle E_j \rangle}. \tag{3.69}$$

Using these approximations, we find that the integral can be approximated in the following way

$$\begin{aligned} &\int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-A(\mathbf{p}-\mathbf{B}_j/A)^2} e^{i\mathbf{p}\cdot\mathbf{L}} e^{-iE_j(\mathbf{p})T} \\ &\approx \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-A(\mathbf{p}-\mathbf{B}_j/A)^2} e^{i\mathbf{p}\cdot\mathbf{L}} e^{-i\langle E_j \rangle T} e^{-i\mathbf{v}_j\cdot\mathbf{p}T} e^{i\mathbf{v}_j\cdot\langle \mathbf{p} \rangle_j T} \\ &= e^{-i(\langle E_j \rangle - \mathbf{v}_j\cdot\langle \mathbf{p} \rangle_j)T} \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-A(\mathbf{p}-\mathbf{B}_j/A)^2} e^{i\mathbf{p}\cdot(\mathbf{L}-\mathbf{v}_j T)} \\ &= \left\{ \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-A(\mathbf{p}-\mathbf{B}_j/A)^2} e^{i\mathbf{p}\cdot(\mathbf{L}-\mathbf{v}_j T)} \right. \\ &= \left. \left(\frac{1}{4\pi A}\right)^{3/2} e^{i[4\mathbf{B}_j\cdot(\mathbf{L}-\mathbf{v}_j T) + i(\mathbf{L}-\mathbf{v}_j T)^2]/(4A)} \right\} \\ &= \left(\frac{1}{4\pi A}\right)^{3/2} e^{-i(\langle E_j \rangle - \mathbf{v}_j\cdot\langle \mathbf{p} \rangle_j)T} e^{i(\mathbf{L}-\mathbf{v}_j T)\cdot\mathbf{B}_j/A} e^{-(\mathbf{L}-\mathbf{v}_j T)^2/(4A)}. \end{aligned} \tag{3.70}$$

Since we are working in the relativistic limit, we have

$$\langle E_j\rangle = \sqrt{\langle \mathbf{p}\rangle_j^2 + m_j^2} \approx |\langle \mathbf{p}\rangle_j| + \frac{m_j^2}{2|\langle \mathbf{p}\rangle_j|}, \tag{3.71}$$

$$\mathbf{v}_j = \frac{\langle \mathbf{p}\rangle_j}{\langle E_j\rangle} \approx \frac{\langle \mathbf{p}\rangle_j}{|\langle \mathbf{p}\rangle_j|} \Rightarrow \mathbf{v}_j \cdot \langle \mathbf{p}\rangle_j = |\langle \mathbf{p}\rangle_j|. \tag{3.72}$$

Using these and the approximate expression for the integral, we obtain

$$\begin{aligned} A^{rs}_{\alpha\beta}(\cdots) &= -\sum_j U_{\beta j}U^*_{\alpha j}\delta^{rs}\left(\frac{1}{4\pi A}\right)^{3/2}\left(\frac{2\pi}{\sigma_p^2}\right)^{3/4}\left(\frac{2\pi}{\sigma_d^2}\right)^{3/4} e^{\mathbf{B}_j^2/A - C_j} \\ &\quad \times e^{-im_j^2T/(2|\langle\mathbf{p}\rangle_j|)} e^{i(\mathbf{L}-\mathbf{v}_jT)\cdot\mathbf{B}_j/A} e^{-(\mathbf{L}-\mathbf{v}_jT)^2/(4A)}. \end{aligned} \tag{3.73}$$

Since we want to investigate how the amplitude depends on $\mathbf{P}_j$, $\mathbf{Q}_j$, σ_p, and σ_d, we use

$$\mathbf{B}_j^2/A - C_j = -\frac{(\mathbf{P}_j - \mathbf{Q}_j)^2}{4(\sigma_p^2+\sigma_d^2)}, \tag{3.74}$$

$$\left(\frac{1}{4\pi A}\right)^{3/2}\left(\frac{2\pi}{\sigma_p^2}\right)^{3/4}\left(\frac{2\pi}{\sigma_d^2}\right)^{3/4} = \left(\frac{2\sigma_p\sigma_d}{\sigma_p^2+\sigma_d^2}\right)^{3/2}, \tag{3.75}$$

$$\langle \mathbf{p}\rangle_j = \mathbf{B}_j/A = \frac{\mathbf{P}_j\sigma_d^2 + \mathbf{Q}_j\sigma_p^2}{\sigma_p^2+\sigma_d^2}. \tag{3.76}$$

Thus, the amplitude in the relativistic approximation is

$$\begin{aligned} A^{rs}_{\alpha\beta}(\cdots) &= -\sum_j U_{\beta j}U^*_{\alpha j}\delta^{rs}\left(\frac{2\sigma_p\sigma_d}{\sigma_p^2+\sigma_d^2}\right)^{3/2} e^{-\frac{(\mathbf{P}_j-\mathbf{Q}_j)^2}{4(\sigma_p^2+\sigma_d^2)}} e^{-im_j^2T\frac{\sigma_p^2+\sigma_d^2}{2|\mathbf{P}_j\sigma_d^2+\mathbf{Q}_j\sigma_p^2|}} \\ &\quad \times e^{i(\mathbf{L}-\hat{\mathbf{n}}_jT)\cdot\frac{\mathbf{P}_j\sigma_d^2+\mathbf{Q}_j\sigma_p^2}{\sigma_p^2+\sigma_d^2}} e^{-(\mathbf{L}-\hat{\mathbf{n}}_jT)^2\frac{\sigma_p^2\sigma_d^2}{\sigma_p^2+\sigma_d^2}}, \end{aligned} \tag{3.77}$$

where $\hat{\mathbf{n}}_j = \frac{\mathbf{P}_j\sigma_d^2+\mathbf{Q}_j\sigma_p^2}{|\mathbf{P}_j\sigma_d^2+\mathbf{Q}_j\sigma_p^2|}$.

The Probability in the Relativistic Limit

The probability that the transition $\alpha \to \beta$ takes place is given by the absolute value squared of the transition amplitude, i.e., $P^{rs}_{\alpha\beta}(\cdots) = |A^{rs}_{\alpha\beta}(\cdots)|^2$, which can be written as

$$\begin{aligned} P^{rs}_{\alpha\beta}(\cdots) &= \sum_{j,k} U_{\beta j}U^*_{\alpha j}U^*_{\beta k}U_{\alpha k}\delta^{rs}\left(\frac{2\sigma_p\sigma_d}{\sigma_p^2+\sigma_d^2}\right)^3 e^{-\frac{(\mathbf{P}_j-\mathbf{Q}_j)^2+(\mathbf{P}_k-\mathbf{Q}_k)^2}{4(\sigma_p^2+\sigma_d^2)}} \\ &\quad \times e^{-i\frac{T(\sigma_p^2+\sigma_d^2)}{2}\left(\frac{m_j^2}{|\mathbf{P}_j\sigma_d^2+\mathbf{Q}_j\sigma_p^2|} - \frac{m_k^2}{|\mathbf{P}_k\sigma_d^2+\mathbf{Q}_k\sigma_p^2|}\right)} \\ &\quad \times e^{\frac{i}{\sigma_p^2+\sigma_d^2}\left[(\mathbf{L}-\hat{\mathbf{n}}_jT)\cdot(\mathbf{P}_j\sigma_d^2+\mathbf{Q}_j\sigma_p^2) - (\mathbf{L}-\hat{\mathbf{n}}_kT)\cdot(\mathbf{P}_k\sigma_d^2+\mathbf{Q}_k\sigma_p^2)\right]} \\ &\quad \times e^{-\left[(\mathbf{L}-\hat{\mathbf{n}}_jT)^2+(\mathbf{L}-\hat{\mathbf{n}}_kT)^2\right]\frac{\sigma_p^2\sigma_d^2}{\sigma_p^2+\sigma_d^2}}, \end{aligned} \tag{3.78}$$

where $\hat{\mathbf{n}}_j = \frac{\mathbf{P}_j\sigma_d^2+\mathbf{Q}_j\sigma_p^2}{|\mathbf{P}_j\sigma_d^2+\mathbf{Q}_j\sigma_p^2|}$.
Some special cases:

1. $\mathbf{P}_j = \mathbf{P}$ and $\mathbf{Q}_j = \mathbf{Q}$ gives

$$\begin{aligned} P^{rs}_{\alpha\beta}(\cdots) &= \sum_{j,k} U_{\beta j}U^*_{\alpha j}U^*_{\beta k}U_{\alpha k}\delta^{rs}\left(\frac{2\sigma_p\sigma_d}{\sigma_p^2+\sigma_d^2}\right)^3 e^{-\frac{(\mathbf{P}-\mathbf{Q})^2}{2(\sigma_p^2+\sigma_d^2)}} \\ &\quad \times e^{-i\frac{T(\sigma_p^2+\sigma_d^2)}{2|\mathbf{P}\sigma_d^2+\mathbf{Q}\sigma_p^2|}\left(m_j^2-m_k^2\right)} e^{-2(\mathbf{L}-\hat{\mathbf{n}}T)^2\frac{\sigma_p^2\sigma_d^2}{\sigma_p^2+\sigma_d^2}}, \end{aligned} \tag{3.79}$$

where $\hat{\mathbf{n}} = \frac{\mathbf{P}\sigma_d^2+\mathbf{Q}\sigma_p^2}{|\mathbf{P}\sigma_d^2+\mathbf{Q}\sigma_p^2|}$.

2. $\mathbf{P}_j = \mathbf{P}$, $\mathbf{Q}_j = \mathbf{Q}$, and $\sigma_p = \sigma_d$ gives

$$\begin{aligned} P^{rs}_{\alpha\beta}(\cdots) &= \sum_{j,k} U_{\beta j}U^*_{\alpha j}U^*_{\beta k}U_{\alpha k}\delta^{rs} e^{-\frac{(\mathbf{P}-\mathbf{Q})^2}{4\sigma_p^2}} \\ &\quad \times e^{-i\frac{T}{|\mathbf{P}+\mathbf{Q}|}\left(m_j^2-m_k^2\right)} e^{-(\mathbf{L}-\hat{\mathbf{n}}T)^2\sigma_p^2}, \end{aligned} \tag{3.80}$$

where $\hat{\mathbf{n}} = \frac{\mathbf{P}+\mathbf{Q}}{|\mathbf{P}+\mathbf{Q}|}$.

Time Average

In practical experiments, the distance $|\mathbf{L}|$ from the neutrino source to the detector is known, whereas the time of propagation T is not measured. This means that the probability at the distance $|\mathbf{L}|$ is given by the time average of the probability expressions given above.

We start by considering case 2, thus

$$\begin{aligned} \overline{P}^{rs}_{\alpha\beta}(\cdots) &= \sum_{j,k} U_{\beta j}U^*_{\alpha j}U^*_{\beta k}U_{\alpha k}\delta^{rs} e^{-\frac{(\mathbf{P}-\mathbf{Q})^2}{4\sigma_p^2}} \\ &\quad \times \frac{\int_{-\infty}^{\infty} e^{-i\frac{T}{|\mathbf{P}+\mathbf{Q}|}\left(m_j^2-m_k^2\right)} e^{-(\mathbf{L}-\hat{\mathbf{n}}T)^2\sigma_p^2} dT}{\int_{-\infty}^{\infty} e^{-(\mathbf{L}-\hat{\mathbf{n}}T)^2\sigma_p^2} dT}. \end{aligned} \tag{3.81}$$

Since

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-i\frac{T}{|\mathbf{P}+\mathbf{Q}|}\left(m_j^2-m_k^2\right)} e^{-(\mathbf{L}-\hat{\mathbf{n}}T)^2\sigma_p^2} dT \\
&= \int_{-\infty}^{\infty} e^{-i\frac{T}{|\mathbf{P}+\mathbf{Q}|}\left(m_j^2-m_k^2\right)} e^{(-\mathbf{L}^2+2\mathbf{L}\cdot\hat{\mathbf{n}}T-T^2)\sigma_p^2} dT \\
&= e^{-\mathbf{L}^2\sigma_p^2} \int_{-\infty}^{\infty} e^{-i\frac{T}{|\mathbf{P}+\mathbf{Q}|}\left(m_j^2-m_k^2\right)} e^{(2\mathbf{L}\cdot\hat{\mathbf{n}}T-T^2)\sigma_p^2} dT \\
&= \frac{\sqrt{\pi}}{\sigma_p} e^{-\mathbf{L}^2\sigma_p^2} e^{\frac{\left[i(m_j^2-m_k^2)-2\sigma_p^2\mathbf{L}\cdot\hat{\mathbf{n}}|\mathbf{P}+\mathbf{Q}|\right]^2}{4\sigma_p^2|\mathbf{P}+\mathbf{Q}|^2}} \\
&= \frac{\sqrt{\pi}}{\sigma_p} e^{-\mathbf{L}^2\sigma_p^2} e^{-\frac{(m_j^2-m_k^2)^2}{4\sigma_p^2|\mathbf{P}+\mathbf{Q}|^2}} e^{-\frac{i(m_j^2-m_k^2)\mathbf{L}\cdot\hat{\mathbf{n}}}{|\mathbf{P}+\mathbf{Q}|}} e^{(\mathbf{L}\cdot\hat{\mathbf{n}})^2\sigma_p^2} \qquad (3.82)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-(\mathbf{L}-\hat{\mathbf{n}}T)^2\sigma_p^2} dT = e^{-\mathbf{L}^2\sigma_p^2} \int_{-\infty}^{\infty} e^{(2\mathbf{L}\cdot\hat{\mathbf{n}}T-T^2)\sigma_p^2} dT \\
&= \frac{\sqrt{\pi}}{\sigma_p} e^{-\mathbf{L}^2\sigma_p^2} e^{\sigma_p^2(\mathbf{L}\cdot\hat{\mathbf{n}})^2}, \qquad (3.83)
\end{aligned}$$

we obtain

$$\begin{aligned}
\overline{P}^{rs}_{\alpha\beta}(\cdots) &= \sum_{j,k} U_{\beta j}U^*_{\alpha j}U^*_{\beta k}U_{\alpha k}\delta^{rs} e^{-\frac{(\mathbf{P}-\mathbf{Q})^2}{4\sigma_p^2}} e^{-\frac{(m_j^2-m_k^2)^2}{4\sigma_p^2|\mathbf{P}+\mathbf{Q}|^2}} \\
&\times e^{-\frac{i(m_j^2-m_k^2)\mathbf{L}\cdot\hat{\mathbf{n}}}{|\mathbf{P}+\mathbf{Q}|}}, \qquad (3.84)
\end{aligned}$$

where $\hat{\mathbf{n}} = \frac{\mathbf{P}+\mathbf{Q}}{|\mathbf{P}+\mathbf{Q}|}$.
Similarly, we have in case 1

$$\begin{aligned}
\overline{P}^{rs}_{\alpha\beta}(\cdots) &= \sum_{j,k} U_{\beta j}U^*_{\alpha j}U^*_{\beta k}U_{\alpha k}\delta^{rs} \left(\frac{2\sigma_p\sigma_d}{\sigma_p^2+\sigma_d^2}\right)^3 e^{-\frac{(\mathbf{P}-\mathbf{Q})^2}{2(\sigma_p^2+\sigma_d^2)}} \\
&\times e^{-\frac{(m_j^2-m_k^2)^2(\sigma_p^2+\sigma_d^2)^3}{32|\mathbf{P}\sigma_d^2+\mathbf{Q}\sigma_p^2|^2\sigma_p^2\sigma_d^2}} e^{-\frac{i(m_j^2-m_k^2)(\sigma_p^2+\sigma_d^2)\mathbf{L}\cdot\hat{\mathbf{n}}}{2|\mathbf{P}\sigma_d^2+\mathbf{Q}\sigma_p^2|}}, \qquad (3.85)
\end{aligned}$$

where $\hat{\mathbf{n}} = \frac{\mathbf{P}\sigma_d^2+\mathbf{Q}\sigma_p^2}{|\mathbf{P}\sigma_d^2+\mathbf{Q}\sigma_p^2|}$.

3.2.5 Comments about the Assumptions in our Treatment

We have considered the neutrino production, mixing, and detection processes as a single process in QFT. In our approach to obtain the neutrino transition probability, we have used the propagators of the neutrino mass eigenfields to connect the source and detection processes.

In a physical situation, a necessary condition for neutrino oscillation to occur is that the neutrino source and detector are localized within a region much smaller than the oscillation length. The localization of the source and detector (and hence a spread of the neutrino momentum due to the uncertainty principle) implies that we need to describe the propagating flavor neutrino by a superposition of localized wave packets. A QM treatment of neutrino oscillations using wave packets has been discussed by Giunti et al. [9].

In deriving our general expressions for the transition amplitude (3.59), we made the following assumptions:

1. the neutrinos are propagating through vacuum,
2. they are produced in a state of definite flavor and spin, and
3. we used a wave packet approach to localize the flavor states.

In order to arrive at the expression (3.78) for the oscillation probability, we had to make some more assumptions:

4. the distribution function $F(\mathbf{X}, \mathbf{P}, T, \mathbf{p})$ is Gaussian, see Eq. (3.61),
5. the neutrinos are extremely relativistic, and
6. each mass eigenstate is centered around a different momentum $\mathbf{P}_j$.

Chapter 4

Neutrinos Propagating through Matter

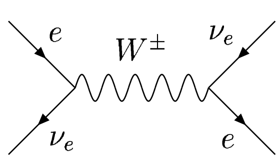
4.1 General Discussion

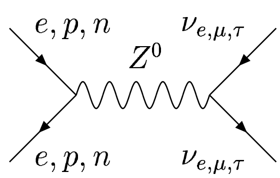
How do we modify the theory in the case of presence of matter?

When a neutrino propagates through a medium the relation between its energy and 3-momentum is not the same as in vacuum. Even if the mass matrix is $\mathcal{CP}$ conserving, matter effects give rise to apparent $\mathcal{CP}$ and $\mathcal{CPT}$ violations.

Normal matter has electrons, but no muons or taus at all at normal temperature. This implies that the ν_e's will interact with matter through both charged- and neutral-current interactions with the electrons. Of course ν_μ and ν_τ will also interact with matter but with a different magnitude than ν_e, since ν_μ or ν_τ interact with the electrons only via the neutral-current. The neutral-current (NC) weak interactions are mediated by Z^0 gauge bosons and the charged-current (CC) weak interactions are mediated by the $W^\pm$ gauge bosons.

The effective mass of a particle is modified by the interactions with the medium through which it is propagating. Since ν_e interacts differently compared with ν_μ or ν_τ the modification for ν_e is different than for the other neutrinos.





Assume a uniform background matter density with n_e, n_p, and n_n denoting the number of electrons, protons and neutrons per unit volume, respectively.

Elastic scattering through CC interactions, $\nu_e + e \to \nu_e + e$, gives a contribution to the effective Lagrangian density which reads

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= -\frac{4G_F}{\sqrt{2}}\left[\overline{e}(p_1)\gamma_\lambda P_L \nu_e(p_2)\right]\left[\overline{\nu}_e(p_3)\gamma^\lambda P_L e(p_4)\right] \\ &= \{\text{Fierz transformation, see [16] pp. 160-162}\} \\ &= -\frac{4G_F}{\sqrt{2}}\left[\overline{\nu}_e(p_3)\gamma_\lambda P_L \nu_e(p_2)\right]\left[\overline{e}(p_1)\gamma^\lambda P_L e(p_4)\right] \end{aligned} \tag{4.1}$$

Here G_F is the Fermi constant. In forward scattering, where $p_2 = p_3 = p$, we have the following contribution, which affects the propagation of ν_e,

$$-\sqrt{2}G_F \overline{\nu}_{eL}(p)\gamma_\lambda \nu_{eL}(p)\left\langle \overline{e}\gamma^\lambda(\mathbb{1}-\gamma^5)e\right\rangle. \tag{4.2}$$

The only non-trivial average is $\langle \overline{e}\gamma^0 e\rangle = \langle e^\dagger e\rangle = n_e$. The derivation of this average can be found in [18, 20, 21]. Thus, the contribution to the effective Lagrangian is $-\sqrt{2}G_F n_e \overline{\nu}_{eL}\gamma^0\nu_{eL}$. This term changes the effective energy of the neutrino to $E = \sqrt{\mathbf{p}^2 + m^2} + \sqrt{2}G_F n_e$, we can interpret this as coming from a potential energy

$$V_{\text{CC}} = \sqrt{2}G_F n_e. \tag{4.3}$$

Similarly, for the NC weak interaction one obtains

$$V_{\text{NC}} = \sqrt{2}G_F \sum_{f=e,p,n} n_f \left[I_{3L}^{(f)} - 2\sin^2\Theta_W Q^{(f)}\right] = \ldots = -\sqrt{2}G_F n_n/2. \tag{4.4}$$

This derivation can be found in [18, 20, 21] and it was first discussed by Wolfenstein [28] in 1978. The NC contribution is the same for all flavors, whereas the CC contribution affects only ν_e.

In general, neutrinos and antineutrinos do not have the same dispersion relation, since the interaction with the electrons are different. But when the background medium is $\mathcal{CP}$ symmetric and the interactions also are $\mathcal{CP}$ invariant, then both ν and $\overline{\nu}$ have the same dispersion relation.

To arrive at the expression $E = \sqrt{\mathbf{p}^2 + m^2} + \sqrt{2}G_F n_e$ one could also proceed in the following way. Define

$$iS_F(k,\mu,\beta) = (\not{k} + m)\left[\frac{i}{\not{k} - m^2 + i\epsilon} - 2\pi\delta(k^2 - m^2)\eta_F(k,\mu,\beta)\right], \tag{4.5}$$

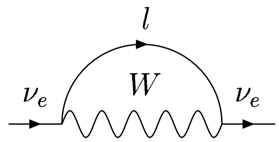
where

$$\eta_F(k,\mu,\beta) = \Theta(k\cdot u)f_F(k,\mu,\beta) + \Theta(-k\cdot u)f_F(-k,-\mu,\beta) \tag{4.6}$$

contains the distribution functions for both fermions and antifermions, and

$$f_F(k,\mu,\beta) = \frac{1}{e^{\beta(k\cdot u-\mu)} + 1}. \tag{4.7}$$

Here μ stands for the chemical potential and $\beta = \frac{1}{k_B T}$ where T is the temperature. This gives that the self-energy arising from the diagram



is

$$-i\sum_a(p) = \left(\frac{ig}{\sqrt{2}}\right)^2 \int \frac{d^4k}{(2\pi)^4}\gamma^\rho P_L i S_e(k,\mu,\beta)\gamma^\lambda P_L i\Delta_{\rho\lambda}(q), \tag{4.8}$$

where $\Delta_{\rho\lambda}$ is the W-propagator, $q = k - p$. We are only interested in the matter-induced self-energy, which is

$$\sum_a{}'(p) = \frac{g^2}{2}\int \frac{d^4k}{(2\pi)^3}\delta(k^2 - m_e^2)\eta_F(k,\mu,\beta)\gamma^\rho \not{k}\gamma^\lambda P_L\Delta_{\rho\lambda}(q). \tag{4.9}$$

Expanding the W-propagator as a power series in $1/M_W^2$, one obtains after some computations that in the rest frame of the medium $E = \sqrt{\mathbf{p}^2 + m^2} + \sqrt{2}G_F n_e$. This has been discussed in [20], which includes all the steps of the computation.

The effective potentials for ν_e and $\overline{\nu}_e$ in different media can be summarized in the following table [18]:

Medium	$V_{\rm CC}$	$V_{\rm NC}$
e^- and e^+	$\pm\sqrt{2}G_F(N_e - N_{\overline{e}})$	$\mp\frac{1}{\sqrt{2}}G_F(N_e - N_{\overline{e}})(1 - 4\sin^2\theta_W)$
p^- and $\overline{p}$	0	$\pm\frac{1}{\sqrt{2}}G_F(N_p - N_{\overline{p}})(1 - 4\sin^2\theta_W)$
n and $\overline{n}$	0	$\mp\frac{1}{\sqrt{2}}G_F(N_n - N_{\overline{n}})$
ν_e and $\overline{\nu}_e$	0	$\pm 2\sqrt{2}G_F(N_{\nu_e} - N_{\overline{\nu}_e})$
$\nu_{\mu,\tau}$ and $\overline{\nu}_{\mu,\tau}$	0	$\pm 2\sqrt{2}G_F(N_{\nu_e} - N_{\overline{\nu}_e})$
e^- and p	$\pm\sqrt{2}G_F N_e$	0
neutral(e^-, n, p)	$\pm\sqrt{2}G_F N_e$	$\mp\frac{1}{\sqrt{2}}G_F N_n$

where upper (lower) signs refer to ν_e ($\overline{\nu}_e$).

The dispersion relation $E_\nu^2 = p^2 + m_\nu^2$ in vacuum changes when a ν_e propagates in matter to

$$\left(E_\nu - \frac{V}{2}\right)^2 = \left(p + \frac{V}{2}\right)^2 + m_\nu^2 \quad \text{for Dirac neutrinos,} \tag{4.10}$$

$$E_\nu^2 = (p + V)^2 + m_\nu^2 \quad \text{for Majorana neutrinos,} \tag{4.11}$$

where the effective potential $V = V_{\rm CC} + V_{\rm NC}$, see [10, 18, 20]. For relativistic Dirac and Majorana neutrinos this reduces to $E_\nu^2 = p^2 + (m_\nu^2 + 2pV) + \cdots$. This can be interpreted in two ways either as $m_{\nu_e}^2 \to m_{\nu_e}^2 + 2pV$ or as $E_\nu^2 \to E_\nu^2 - 2pV$.

4.2 Neutrino Oscillations in a Uniform Background

4.2.1 Quantum Field Theoretical Treatment

In chapter 3.2, we studied neutrino oscillations in vacuum. As we have already discussed neutrinos are affected by the presence of matter. We are here going to treat the very simplest case, namely neutrinos propagating through a static uniform background and see how this affects the probability oscillation formula. The Feynman fermion propagator in vacuum is

$$S_j(y'-x') = \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k}+m_j)}{k^2-m_j^2+i\epsilon} e^{-ik(y'-x')}. \tag{4.12}$$

The factor $\frac{i(\not{k}+m_j)}{k^2-m_j^2}$ can be written as

$$\frac{i\,(\not{k}+m_j)}{k^2-m_j^2} = \frac{i\,(\not{k}+m_j)}{(\not{k}+m_j)(\not{k}-m_j)} = \frac{i}{\not{k}-m_j}. \tag{4.13}$$

The factor $\frac{i}{\not{k}-m_j}$ corresponds to a neutrino propagating without interacting with matter. When neutrinos propagate through matter the Feynman fermion propagator is modified according to

$$S_j(y'-x') \to S_{jj}(y'-x') + S_{jk}(y'-x'), \tag{4.14}$$

where S_{jj} and S_{jk} are the diagonal and off-diagonal parts, respectively. This is due to the fact that we cannot make use of Eq. (3.50) in Eq. (3.48), since the mass eigenstate k can propagate into a different mass eigenstate j, because of the interaction with matter. Thus we have to use

$$\begin{aligned} \langle 0|\nu_j^s(y')\nu_k^{r\dagger}(x')|0\rangle &\to \langle 0|T(\nu_j^s(y')\nu_k^{r\dagger}(x'))|0\rangle \\ &= S_{jj}(y'-x')\gamma^0 + S_{jk}(y'-x')\gamma^0 \end{aligned} \tag{4.15}$$

in Eq. (3.48). The method we are going to use to compute S_{jj} and S_{jk} is rather simple since, we will only include the first non-trivial contribution, i.e., when the neutrinos interact only once with matter. However, the correct way would be to sum up all different contributions. We can therefore not expect to recover the QM result. An alternative way to find the propagator is the following. Adding the contribution $\sum_{j,k} \overline{\nu}_j \not{A}_{jk} \nu_k$ (see Eq. (4.49)) to the interaction Lagrangian (3.34) and then using the Euler-Lagrange equation of motion, we find that Eq. (3.38) valid in vacuum is modified to

$$(i\not{\partial} - m_j)\nu_j(x) + \sum_k \not{A}_{jk}\nu_k(x) = \chi_j(x), \tag{4.16}$$

where the source term $\chi_j(x)$ is still given by Eq. (3.39). To zeroth order in the coupling constant g ($\chi_j(x)$ depends on g), we have that $\chi_j(x) = 0$ and thus the equation of motion becomes

$$(i\partial\!\!\!/ - m_j)\nu_j(x) + \sum_k A\!\!\!/_{jk}\nu_k(x) = 0. \tag{4.17}$$

If we could solve this system of PDEs, we could easily compute

$$\langle 0|T(\nu_j^s(y')\nu_k^{r\dagger}(x'))|0\rangle. \tag{4.18}$$

Then one should be able to compute the expression for the amplitude (3.48), at least in the relativistic limit. But we will not follow this approach, since the problem of solving Eq. (4.17) is very difficult. Instead we will use S_{jj} and S_{jk}. We start out by computing S_{jj} and its contribution to the amplitude. If the neutrino interacts once with the matter, we obtain a term $\frac{i}{k\!\!\!/-m_j} iA\!\!\!/_{jj} \frac{i}{k\!\!\!/-m_j}$. We are then interested in the expression

$$G = \frac{i}{k\!\!\!/ - m_j} + \frac{i}{k\!\!\!/ - m_j} iA\!\!\!/_{jj} \frac{i}{k\!\!\!/ - m_j} + \frac{i}{k\!\!\!/ - m_j} iA\!\!\!/_{jj} \frac{i}{k\!\!\!/ - m_j} iA\!\!\!/_{jj} \frac{i}{k\!\!\!/ - m_j} + \cdots \tag{4.19}$$

i.e., for the case the neutrino interacts zero, one, two, or more times with the matter, but without changing the mass. Using the identity

$$\frac{1}{A-B} = \frac{1}{A} + \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \cdots \tag{4.20}$$

which holds for any two operators A and B even though they do not commute, we have

$$G = \frac{i}{k\!\!\!/ + A\!\!\!/_{jj} - m_j}. \tag{4.21}$$

We can rewrite this expression as follows

$$G = \frac{i}{k\!\!\!/ + A\!\!\!/_{jj} - m_j} = \frac{i\,(k\!\!\!/ + A\!\!\!/_{jj} + m_j)}{(k\!\!\!/ + A\!\!\!/_{jj} - m_j)(k\!\!\!/ + A\!\!\!/_{jj} + m_j)} = \frac{i\,(k\!\!\!/ + A\!\!\!/_{jj} + m_j)}{k^2 + 2kA_{jj} + A_{jj}^2 - m_j^2}. \tag{4.22}$$

We thus have

$$S_{jj}(y' - x') = \int \frac{d^4k}{(2\pi)^4} \frac{i(k\!\!\!/ + A\!\!\!/_{jj} + m_j)}{(k + A_{jj})^2 - m_j^2 + i\epsilon} e^{-ik(y'-x')}. \tag{4.23}$$

The contribution from the diagonal part to the amplitude then becomes

$$
\begin{aligned}
A^{rs}_{\alpha\beta}(\cdots)^{\text{diagonal}}_{\text{matter}} &= \sum_j U_{\beta j} U^*_{\alpha j} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{F^\dagger(\mathbf{Y}, \mathbf{Q}, T_d, \mathbf{q})}{\sqrt{2E_\beta(\mathbf{q})}} \\
&\times \int d^3\mathbf{y}' e^{iqy'} u^{s\dagger}_j(q) \cdot \\
&\times \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + \not{A}_{jj} + m_j)}{(k + A_{jj})^2 - m_j^2 + i\epsilon} e^{-ik(y'-x')} \gamma^0 \\
&\times \int d^3\mathbf{x}' e^{-ipx'} u^r_j(p) \\
&\times \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{F(\mathbf{X}, \mathbf{P}, T_p, \mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}}. \qquad (4.24)
\end{aligned}
$$

If we make the change of variable $z_{jj} = k + A_{jj}$ and proceed in the same way as was done in Eqs. (3.53)-(3.59), we have

$$
\begin{aligned}
A^{rs}_{\alpha\beta}(\cdots)^{\text{diagonal}}_{\text{matter}} &= -\sum_j U_{\beta j} U^*_{\alpha j} \delta^{rs} e^{iA_{jj}(y'-x')} \\
&\times \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{F^\dagger(\mathbf{Y}, \mathbf{Q}, T_d, \mathbf{p})}{\sqrt{2E_\beta(\mathbf{p})}} \frac{F(\mathbf{X}, \mathbf{P}, T_p, \mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}} 2E_j(\mathbf{p}).
\end{aligned}
\qquad (4.25)
$$

We continue with the off-diagonal part S_{jk} and its contribution to the amplitude. We are now interested in terms such as

$$
\frac{i}{\not{k} - m_j} i\not{A}_{jk} \frac{i}{\not{k} - m_k} \qquad (4.26)
$$

and so on, when $j \neq k$. For simplicity we will only study this term and thus make the approximation

$$
S_{jk}(y' - x') \approx \int \frac{d^4k}{(2\pi)^4} \frac{i}{\not{k} - m_j} i\not{A}_{jk} \frac{i}{\not{k} - m_k} e^{-ik(y'-x')}. \qquad (4.27)
$$

The contribution from the off-diagonal part to the amplitude then becomes

$$
\begin{aligned}
A^{rs}_{\alpha\beta}(\cdots)^{\text{off-diagonal}}_{\text{matter}} &= -i \sum_{j \neq k} U_{\beta j} U^*_{\alpha k} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{F^\dagger(\mathbf{Y}, \mathbf{Q}, T_d, \mathbf{q})}{\sqrt{2E_\beta(\mathbf{q})}} \\
&\times \int d^3\mathbf{y}' e^{iqy'} u^{s\dagger}_j(q) \\
&\times \int \frac{d^4k}{(2\pi)^4} \frac{1}{\not{k} - m_j} \not{A}_{jk} \frac{1}{\not{k} - m_k} e^{-ik(y'-x')} \gamma^0 \\
&\times \int d^3\mathbf{x}' e^{-ipx'} u^r_k(p) \\
&\times \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{F(\mathbf{X}, \mathbf{P}, T_p, \mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}}. \qquad (4.28)
\end{aligned}
$$

The three integrals in the middle can be calculated in a straightforward manner. If we introduce the notation $\not{p}_j \equiv \gamma^0 E_j(\mathbf{p}) + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3$, we have from appendix C that

$$\begin{aligned}
&\int d^3\mathbf{y}' e^{iqy'} \int_{C_F} \frac{d^4k}{(2\pi)^4} \frac{1}{\not{k} - m_j} \not{A}_{jk} \frac{1}{\not{k} - m_k} e^{-ik(y'-x')} \int d^3\mathbf{x}' e^{-ipx'} \\
&= (2\pi)^2 \delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ix'^0 E_k(\mathbf{p})} e^{iy'^0 E_j(\mathbf{q})} 2\pi i \\
&\times \left[\left(\frac{(\not{p}_j + m_j) \not{A}_{jk} (\not{p}_j + m_k)}{2E_j(\mathbf{p}) \left(E_j^2(\mathbf{p}) - E_k^2(\mathbf{p})\right)} \right) e^{-iE_j(\mathbf{p})(y'^0 - x'^0)} \right. \\
&+ \left. \left(\frac{(\not{p}_k + m_j) \not{A}_{jk} (\not{p}_k + m_k)}{2E_k(\mathbf{p}) \left(E_k^2(\mathbf{p}) - E_j^2(\mathbf{p})\right)} \right) e^{-iE_k(\mathbf{p})(y'^0 - x'^0)} \right]. \qquad (4.29)
\end{aligned}$$

Using this gives

$$\begin{aligned}
&A^{rs}_{\alpha\beta}(\cdots)^{\text{off-diagonal}}_{\text{matter}} \\
&= \sum_{j \neq k} U_{\beta j} U^*_{\alpha k} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{F^\dagger(\mathbf{Y}, \mathbf{Q}, T_d, \mathbf{q})}{\sqrt{2E_\beta(\mathbf{q})}} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{F(\mathbf{X}, \mathbf{P}, T_p, \mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}} \\
&\times (2\pi)^2 \delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ix'^0 E_k(\mathbf{p})} e^{iy'^0 E_j(\mathbf{q})} 2\pi \\
&\times u_j^{s\dagger}(q) \left[\left(\frac{(\not{p}_j + m_j) \not{A}_{jk} (\not{p}_j + m_k)}{2E_j(\mathbf{p}) \left(E_j^2(\mathbf{p}) - E_k^2(\mathbf{p})\right)} \right) e^{-iE_j(\mathbf{p})(y'^0 - x'^0)} \right. \\
&+ \left. \left(\frac{(\not{p}_k + m_j) \not{A}_{jk} (\not{p}_k + m_k)}{2E_k(\mathbf{p}) \left(E_k^2(\mathbf{p}) - E_j^2(\mathbf{p})\right)} \right) e^{-iE_k(\mathbf{p})(y'^0 - x'^0)} \right] \gamma^0 u_k^r(p).
\end{aligned} \qquad (4.30)$$

We can now easily perform the integration over $\mathbf{q}$. If we also use that $\not{A}_{jk} = \gamma^0 A_{jk}$, we have

$$\begin{aligned}
&A^{rs}_{\alpha\beta}(\cdots)^{\text{off-diagonal}}_{\text{matter}} \\
&= \sum_{j \neq k} U_{\beta j} U^*_{\alpha k} \frac{\pi}{(2\pi)^4} \int d^3\mathbf{p} \frac{F^\dagger(\mathbf{Y}, \mathbf{Q}, T_d, \mathbf{p})}{\sqrt{2E_\beta(\mathbf{p})}} \frac{F(\mathbf{X}, \mathbf{P}, T_p, \mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}} \\
&\times e^{-ix'^0 E_k(\mathbf{p})} e^{iy'^0 E_j(\mathbf{p})} A_{jk} \\
&\times u_j^{s\dagger}(E_j(\mathbf{p}), \mathbf{p}) \left[\left(\frac{(\not{p}_j + m_j) \gamma^0 (\not{p}_j + m_k)}{E_j(\mathbf{p}) \left(E_j^2(\mathbf{p}) - E_k^2(\mathbf{p})\right)} \right) e^{-iE_j(\mathbf{p})(y'^0 - x'^0)} \right. \\
&+ \left. \left(\frac{(\not{p}_k + m_j) \gamma^0 (\not{p}_k + m_k)}{E_k(\mathbf{p}) \left(E_k^2(\mathbf{p}) - E_j^2(\mathbf{p})\right)} \right) e^{-iE_k(\mathbf{p})(y'^0 - x'^0)} \right] \gamma^0 u_k^r(E_k(\mathbf{p}), \mathbf{p}).
\end{aligned} \qquad (4.31)$$

We see that we need to evaluate some expressions involving the spinors. This is done in a straightforward way, we have

$$
\begin{aligned}
& u_j^{s\dagger}(E_j(\mathbf{p}),\mathbf{p})\,(\not{p}_j + m_j)\,\gamma^0\,(\not{p}_j + m_k)\,\gamma^0 u_k^r(E_k(\mathbf{p}),\mathbf{p}) \\
&= u_j^{s\dagger}(E_j(\mathbf{p}),\mathbf{p}) \sum_k u_j^k(p) u_j^{k\dagger}(p) \underbrace{\gamma^0\gamma^0}_{=\mathbb{1}} (\not{p}_j + m_k)\,\gamma^0 u_k^r(E_k(\mathbf{p}),\mathbf{p}) \\
&= \sum_k \underbrace{u_j^{s\dagger}(E_j(\mathbf{p}),\mathbf{p}) u_j^k(p)}_{=2E_j(\mathbf{p})\delta^{sk}} u_j^{k\dagger}(p)\,(\not{p}_j + m_k)\,\gamma^0 u_k^r(E_k(\mathbf{p}),\mathbf{p}) \\
&= 2E_j(\mathbf{p}) u_j^{s\dagger}(p)\,(\not{p}_j + m_k)\,\gamma^0 u_k^r(E_k(\mathbf{p}),\mathbf{p}) \\
&= 2E_j(\mathbf{p}) u_j^{s\dagger}(p)\,(\not{p}_j - \not{p}_k + \not{p}_k + m_k)\,\gamma^0 u_k^r(E_k(\mathbf{p}),\mathbf{p}) \\
&= 2E_j(\mathbf{p}) u_j^{s\dagger}(p) \Bigg[\gamma^0\,(E_j(\mathbf{p}) - E_k(\mathbf{p})) \\
&+ \sum_s u_k^s(p) u_k^{s\dagger}(p)\gamma^0 \Bigg]\, \gamma^0 u_k^r(E_k(\mathbf{p}),\mathbf{p}) \\
&= 2E_j(\mathbf{p}) u_j^{s\dagger}(p)\,(E_j(\mathbf{p}) - E_k(\mathbf{p}) + 2E_k(\mathbf{p}))\, u_k^r(p) \\
&= 2E_j(\mathbf{p})\,(E_j(\mathbf{p}) + E_k(\mathbf{p}))\, u_j^{s\dagger}(p) u_k^r(p).
\end{aligned}
\tag{4.32}
$$

In the same way, we have

$$
\begin{aligned}
& u_j^{s\dagger}(E_j(\mathbf{p}),\mathbf{p})\,(\not{p}_k + m_j)\,\gamma^0\,(\not{p}_k + m_k)\,\gamma^0 u_k^r(E_k(\mathbf{p}),\mathbf{p}) \\
&= 2E_k(\mathbf{p})\,(E_j(\mathbf{p}) + E_k(\mathbf{p}))\, u_j^{s\dagger}(p) u_k^r(p).
\end{aligned}
\tag{4.33}
$$

Using this yields

$$
\begin{aligned}
& A_{\alpha\beta}^{rs}(\cdots)_{\text{matter}}^{\text{off-diagonal}} \\
&= \sum_{j\neq k} U_{\beta j} U_{\alpha k}^* \frac{A_{jk}}{(2\pi)^3} \int d^3\mathbf{p}\, \frac{F^\dagger(\mathbf{Y},\mathbf{Q},T_d,\mathbf{p})}{\sqrt{2E_\beta(\mathbf{p})}} \frac{F(\mathbf{X},\mathbf{P},T_p,\mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}} \\
&\times \frac{1}{E_j(\mathbf{p}) - E_k(\mathbf{p})} \left[e^{ix'^0(E_j(\mathbf{p})-E_k(\mathbf{p}))} - e^{iy'^0(E_j(\mathbf{p})-E_k(\mathbf{p}))} \right] u_j^{s\dagger}(p) u_k^r(p).
\end{aligned}
\tag{4.34}
$$

In order to proceed we have to compute $u_j^{s\dagger}(p)u_k^r(p) = u_j^{s\dagger}(E_j(\mathbf{p}),\mathbf{p})u_k^r(E_k(\mathbf{p}),\mathbf{p})$. Let ξ^r satisfy $(\xi^r)^\dagger \xi^s = \delta^{rs}$. We can then write

$$
u_k^r(p) = \sqrt{E_k(\mathbf{p}) + m_k} \begin{pmatrix} \xi^r \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_k(\mathbf{p})+m_k}\xi^r \end{pmatrix}.
\tag{4.35}
$$

Using this we have

$$
\begin{aligned}
&u_j^{s\dagger}(p)u_k^r(p) = \sqrt{E_j(\mathbf{p})+m_j}\left((\xi^s)^\dagger \quad (\xi^s)^\dagger \tfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_j(\mathbf{p})+m_j}\right) \\
&\times\sqrt{E_k(\mathbf{p})+m_k}\begin{pmatrix}\xi^r \\ \tfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_k(\mathbf{p})+m_k}\xi^r\end{pmatrix} \\
&= \sqrt{(E_j(\mathbf{p})+m_j)(E_k(\mathbf{p})+m_k)} \\
&\times\left((\xi^s)^\dagger\xi^r + (\xi^s)^\dagger \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2}{(E_j(\mathbf{p})+m_j)(E_k(\mathbf{p})+m_k)}\xi^r\right) \\
&= \left\{(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 = \mathbf{p}^2\mathbb{1}_{2\times 2}\right\} \\
&= \sqrt{(E_j(\mathbf{p})+m_j)(E_k(\mathbf{p})+m_k)}\left(1+\frac{\mathbf{p}^2}{(E_j(\mathbf{p})+m_j)(E_k(\mathbf{p})+m_k)}\right)\delta^{rs} \\
&= \frac{\delta^{rs}}{\sqrt{(E_j(\mathbf{p})+m_j)(E_k(\mathbf{p})+m_k)}}\left((E_j(\mathbf{p})+m_j)(E_k(\mathbf{p})+m_k)+\mathbf{p}^2\right) \\
&\approx 2E_j(\mathbf{p})\delta^{rs} \approx 2E_k(\mathbf{p})\delta^{rs},
\end{aligned}
\tag{4.36}
$$

where the last approximations are valid in the relativistic limit. Thus, in the relativistic limit, we have

$$
\begin{aligned}
&A_{\alpha\beta}^{rs}(\cdots)_{\text{matter}}^{\text{off-diagonal}} \\
&= \sum_{j\neq k} U_{\beta j}U_{\alpha k}^{*}\frac{A_{jk}\delta^{rs}}{(2\pi)^3}\int d^3\mathbf{p}F^\dagger(\mathbf{Y},\mathbf{Q},T_d,\mathbf{p})F(\mathbf{X},\mathbf{P},T_p,\mathbf{p}) \\
&\times\frac{1}{E_j(\mathbf{p})-E_k(\mathbf{p})}\left[e^{ix'^0(E_j(\mathbf{p})-E_k(\mathbf{p}))} - e^{iy'^0(E_j(\mathbf{p})-E_k(\mathbf{p}))}\right].
\end{aligned}
\tag{4.37}
$$

We are now going to make the same assumptions as in section 3.2.4. Then Eq. (4.25) can be written as

$$
\begin{aligned}
A_{\alpha\beta}^{rs}(\cdots)_{\text{matter}}^{\text{diagonal}} &= -\sum_j U_{\beta j}U_{\alpha j}^{*}\delta^{rs}\left(\frac{2\sigma_p\sigma_d}{\sigma_p^2+\sigma_d^2}\right)^{3/2} e^{-\frac{(\mathbf{P}_j-\mathbf{Q}_j)^2}{4(\sigma_p^2+\sigma_d^2)}} \\
&\times e^{-im_j^2T\frac{\sigma_p^2+\sigma_d^2}{2|\mathbf{P}_j\sigma_d^2+\mathbf{Q}_j\sigma_p^2|}} e^{i(\mathbf{L}-\hat{\mathbf{n}}_jT)\cdot\frac{\mathbf{P}_j\sigma_d^2+\mathbf{Q}_j\sigma_p^2}{\sigma_p^2+\sigma_d^2}} \\
&\times e^{-(\mathbf{L}-\hat{\mathbf{n}}_jT)^2\frac{\sigma_p^2\sigma_d^2}{\sigma_p^2+\sigma_d^2}} e^{iA_{jj}(y'-x')},
\end{aligned}
\tag{4.38}
$$

where $\hat{\mathbf{n}}_j = \frac{\mathbf{P}_j\sigma_d^2+\mathbf{Q}_j\sigma_p^2}{|\mathbf{P}_j\sigma_d^2+\mathbf{Q}_j\sigma_p^2|}$. We also see that Eq. (4.37) can be written as

$$\begin{aligned} A^{rs}_{\alpha\beta}(\cdots)^{\text{off-diagonal}}_{\text{matter}} &= \sum_{j\neq k} U_{\beta j}U^*_{\alpha k}A_{jk}\delta^{rs}\left(\frac{2\pi}{\sigma_p^2}\right)^{3/4}\left(\frac{2\pi}{\sigma_d^2}\right)^{3/4} e^{\mathbf{B}_j^2/A-C_j} \\ &\times \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-A(\mathbf{p}-\mathbf{B}_j/A)^2} e^{i\mathbf{p}\cdot\mathbf{L}} e^{-iE_j(\mathbf{p})T} \\ &\times \frac{1}{E_j(\mathbf{p})-E_k(\mathbf{p})}\left[e^{ix'^0(E_j(\mathbf{p})-E_k(\mathbf{p}))} - e^{iy'^0(E_j(\mathbf{p})-E_k(\mathbf{p}))}\right], \end{aligned} \tag{4.39}$$

where the notation is the same as in section 3.2.4. To simplify things we are going to study the case $\mathbf{P}_j = \mathbf{Q}_j = \mathbf{P}$ and $\sigma_p = \sigma_d = \sigma$. We thus have

$$\begin{aligned} A^{rs}_{\alpha\beta}(\cdots)^{\text{diagonal}}_{\text{matter}} &= -\sum_j U_{\beta j}U^*_{\alpha j}\delta^{rs} e^{-\frac{im_j^2T}{2|\mathbf{P}|}} e^{i(\mathbf{L}-\hat{\mathbf{n}}T)\cdot\mathbf{P}} \\ &\times e^{-(\mathbf{L}-\hat{\mathbf{n}}T)^2\sigma^2/2} e^{iA_{jj}(y'-x')} \end{aligned} \tag{4.40}$$

and

$$\begin{aligned} A^{rs}_{\alpha\beta}(\cdots)^{\text{off-diagonal}}_{\text{matter}} &= \sum_{j\neq k} U_{\beta j}U^*_{\alpha k}A_{jk}\delta^{rs}\left(\frac{2\pi}{\sigma^2}\right)^{3/2} \\ &\times \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-(\mathbf{p}-\mathbf{P})^2/(2\sigma^2)} e^{i\mathbf{p}\cdot\mathbf{L}} e^{-iE_j(\mathbf{p})T} \\ &\times \frac{1}{E_j(\mathbf{p})-E_k(\mathbf{p})}\left[e^{ix'^0(E_j(\mathbf{p})-E_k(\mathbf{p}))} - e^{iy'^0(E_j(\mathbf{p})-E_k(\mathbf{p}))}\right], \end{aligned} \tag{4.41}$$

where $\hat{\mathbf{n}} = \mathbf{P}/|\mathbf{P}|$. To compute the integral in Eq. (4.41), we follow the same steps as in the vacuum case and use the approximations (3.67)-(3.72). We thus have

$$\begin{aligned} A^{rs}_{\alpha\beta}(\cdots)^{\text{off-diagonal}}_{\text{matter}} &= \sum_{j\neq k} U_{\beta j}U^*_{\alpha k}\frac{2|\mathbf{P}|A_{jk}\delta^{rs}}{m_j^2-m_k^2} e^{-\frac{im_j^2T}{2|\mathbf{P}|}} e^{i(\mathbf{L}-\hat{\mathbf{n}}T)\cdot\mathbf{P}} e^{-(\mathbf{L}-\hat{\mathbf{n}}T)^2\sigma^2/2} \\ &\times \left[e^{ix'^0\frac{m_j^2-m_k^2}{2|\mathbf{P}|}} - e^{iy'^0\frac{m_j^2-m_k^2}{2|\mathbf{P}|}}\right], \end{aligned} \tag{4.42}$$

where $\hat{\mathbf{n}} = \mathbf{P}/|\mathbf{P}|$. When we computed $A^{rs}_{\alpha\beta}(\cdots)^{\text{diagonal}}_{\text{matter}}$ we managed to sum up many terms, but when we computed $A^{rs}_{\alpha\beta}(\cdots)^{\text{off-diagonal}}_{\text{matter}}$ we just used the first term. Thus, to compute the amplitude, $A^{rs}_{\alpha\beta}(\cdots)_{\text{matter}} = A^{rs}_{\alpha\beta}(\cdots)^{\text{diagonal}}_{\text{matter}} +$

$A^{rs}_{\alpha\beta}(\cdots)^{\text{off-diagonal}}_{\text{matter}}$, to first order in the interaction with matter we must expand $e^{iA_{jj}(y'-x')}$ in Eq. (4.40) and just keep the first two terms. Using this gives

$$\begin{aligned} A^{rs}_{\alpha\beta}(\cdots)_{\text{matter}} &= \Bigg[-\sum_j U_{\beta j}U^*_{\alpha j}\left[1+iA_{jj}(y'^0-x'^0)\right] \\ &\quad + \sum_{j\neq k} U_{\beta j}U^*_{\alpha k}\frac{2|\mathbf{P}|A_{jk}}{m_j^2-m_k^2}\left(e^{ix'^0\frac{m_j^2-m_k^2}{2|\mathbf{P}|}}-e^{iy'^0\frac{m_j^2-m_k^2}{2|\mathbf{P}|}}\right)\Bigg] \\ &\quad \times\delta^{rs}e^{-\frac{im_j^2T}{2|\mathbf{P}|}}e^{i(\mathbf{L}-\hat{\mathbf{n}}T)\cdot\mathbf{P}}e^{-(\mathbf{L}-\hat{\mathbf{n}}T)^2\sigma^2/2} \\ &= \sum_{j,k} U_{\beta j}U^*_{\alpha k}\left[-\delta^{jk}+\frac{2|\mathbf{P}|A_{jk}}{m_j^2-m_k^2}\left(e^{ix'^0\frac{m_j^2-m_k^2}{2|\mathbf{P}|}}-e^{iy'^0\frac{m_j^2-m_k^2}{2|\mathbf{P}|}}\right)\right] \\ &\quad \times\delta^{rs}e^{-\frac{im_j^2T}{2|\mathbf{P}|}}e^{i(\mathbf{L}-\hat{\mathbf{n}}T)\cdot\mathbf{P}}e^{-(\mathbf{L}-\hat{\mathbf{n}}T)^2\sigma^2/2}, \end{aligned} \tag{4.43}$$

where $\hat{\mathbf{n}}=\mathbf{P}/|\mathbf{P}|$. The first term δ^{jk} corresponds to the neutrino propagating through vacuum (not interacting with matter), while the second term corresponds to the neutrino interacting once with matter. The probability that the transition $\alpha\to\beta$ takes place is given by $P^{rs}_{\alpha\beta}(\cdots)_{\text{matter}}=|A^{rs}_{\alpha\beta}(\cdots)_{\text{matter}}|^2$. We thus have

$$\begin{aligned} P^{rs}_{\alpha\beta}(\cdots)_{\text{matter}} &= N\sum_{j,k}\sum_{j',k'} U_{\beta j}U^*_{\alpha k}U^*_{\beta j'}U_{\alpha k'} \\ &\quad\times\left[-\delta^{jk}+\frac{2|\mathbf{P}|A_{jk}}{m_j^2-m_k^2}\left(e^{ix'^0\frac{m_j^2-m_k^2}{2|\mathbf{P}|}}-e^{iy'^0\frac{m_j^2-m_k^2}{2|\mathbf{P}|}}\right)\right] \\ &\quad\times\left[-\delta^{j'k'}+\frac{2|\mathbf{P}|A_{j'k'}}{m_{j'}^2-m_{k'}^2}\left(e^{ix'^0\frac{m_{j'}^2-m_{k'}^2}{2|\mathbf{P}|}}-e^{iy'^0\frac{m_{j'}^2-m_{k'}^2}{2|\mathbf{P}|}}\right)\right]^* \\ &\quad\times\delta^{rs}e^{-\frac{i\left(m_j^2-m_{j'}^2\right)T}{2|\mathbf{P}|}}e^{-(\mathbf{L}-\hat{\mathbf{n}}T)^2\sigma^2}, \end{aligned} \tag{4.44}$$

where $\hat{\mathbf{n}}=\mathbf{P}/|\mathbf{P}|$. The factor N is a normalization constant put in by hand that one has to determine from the condition $\sum_\beta P^{rs}_{\alpha\beta}(\cdots)_{\text{matter}}=1$. Since we have determined the amplitude to first order in A_{jk} we are going to do the same with the probability, we thus have

$$\begin{aligned} P^{rs}_{\alpha\beta}(\cdots)_{\text{matter}} &= N \sum_{j,k} \sum_{j',k'} U_{\beta j} U^*_{\alpha k} U^*_{\beta j'} U_{\alpha k'} \Bigg[\delta^{jk} \delta^{j'k'} \\ &- \delta^{jk} \frac{2|\mathbf{P}| A^*_{j'k'}}{m^2_{j'} - m^2_{k'}} \left(e^{-ix'^0 \frac{m^2_{j'} - m^2_{k'}}{2|\mathbf{P}|}} - e^{-iy'^0 \frac{m^2_{j'} - m^2_{k'}}{2|\mathbf{P}|}} \right) \\ &- \delta^{j'k'} \frac{2|\mathbf{P}| A_{jk}}{m^2_j - m^2_k} \left(e^{ix'^0 \frac{m^2_j - m^2_k}{2|\mathbf{P}|}} - e^{iy'^0 \frac{m^2_j - m^2_k}{2|\mathbf{P}|}} \right) \Bigg] \\ &\times \delta^{rs} e^{-\frac{i\left(m^2_j - m^2_{j'}\right)T}{2|\mathbf{P}|}} e^{-(\mathbf{L} - \hat{\mathbf{n}}T)^2 \sigma^2}. \end{aligned} \tag{4.45}$$

This can be rewritten as

$$\begin{aligned} &P^{rs}_{\alpha\beta}(\cdots)_{\text{matter}} = N \sum_{j} \sum_{j',k'} \Bigg[U_{\beta j} U^*_{\alpha j} U^*_{\beta j'} U_{\alpha k'} \delta^{j'k'} \\ &-2\Re \left(U_{\beta j} U^*_{\alpha j} U^*_{\beta j'} U_{\alpha k'} \frac{2|\mathbf{P}| A^*_{j'k'}}{m^2_{j'} - m^2_{k'}} \left(e^{-ix'^0 \frac{m^2_{j'} - m^2_{k'}}{2|\mathbf{P}|}} - e^{-iy'^0 \frac{m^2_{j'} - m^2_{k'}}{2|\mathbf{P}|}} \right) \right) \Bigg] \\ &\times \delta^{rs} e^{-\frac{i\left(m^2_j - m^2_{j'}\right)T}{2|\mathbf{P}|}} e^{-(\mathbf{L} - \hat{\mathbf{n}}T)^2 \sigma^2}. \end{aligned} \tag{4.46}$$

This is our final result, and we will not try to go any further by using an explicit representation of the matrix U. If one would like to proceed, one would need an expression for A_{jk}. We would like to have an expression for A_{jk} in the case of three neutrino flavors, but since it would be very complicated we will specialize to the case of two neutrino flavors e and μ. In the flavor basis, the matter potential V_f is diagonal

$$V_f = \begin{pmatrix} A_C + A_N & 0 \\ 0 & A_N \end{pmatrix}, \tag{4.47}$$

where $A_C = \sqrt{2} G_F N_e$ and $A_N = -\sqrt{2} G_F N_n$. The unitary mixing matrix U for two flavors can be written as

$$U = \begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.48}$$

The contribution to the effective Lagrangian is

$$\begin{aligned} &-\overline{\nu}_{eL}(A_C + A_N)\gamma^0 \nu_{eL} - \overline{\nu}_{\mu L} A_N \gamma^0 \nu_{\mu L} \equiv \overline{\nu}_{eL} \not{B} \nu_{eL} + \overline{\nu}_{\mu L} \not{C} \nu_{\mu L} \\ &\equiv \sum_{j,k} \overline{\nu}_j \not{A}_{jk} \nu_k. \end{aligned} \tag{4.49}$$

Since $\nu_\alpha = \sum_j U_{\alpha j}\nu_j$ and $\overline{\nu}_\alpha = \sum_j U^*_{\alpha j}\overline{\nu}_j$, we have

$$\begin{aligned}\overline{\nu}_{eL}\not{B}\nu_{eL} &= \sum_j U^*_{ej}\overline{\nu}_j \not{B} \sum_k U_{ek}\nu_k \\ &= \overline{\nu}_1 U^*_{e1}\not{B}U_{e1}\nu_1 + \overline{\nu}_1 U^*_{e1}\not{B}U_{e2}\nu_2 + \overline{\nu}_2 U^*_{e2}\not{B}U_{e1}\nu_1 + \overline{\nu}_2 U^*_{e2}\not{B}U_{e2}\nu_2\end{aligned} \tag{4.50}$$

and

$$\begin{aligned}&\overline{\nu}_{\mu L}\not{C}\nu_{\mu L} \\ &= \overline{\nu}_1 U^*_{\mu 1}\not{C}U_{\mu 1}\nu_1 + \overline{\nu}_1 U^*_{\mu 1}\not{C}U_{\mu 2}\nu_2 + \overline{\nu}_2 U^*_{\mu 2}\not{C}U_{\mu 1}\nu_1 + \overline{\nu}_2 U^*_{\mu 2}\not{C}U_{\mu 2}\nu_2.\end{aligned} \tag{4.51}$$

Thus,

$$\begin{aligned}\not{A}_{11} &= U^*_{e1}\not{B}U_{e1} + U^*_{\mu 1}\not{C}U_{\mu 1} = -\left(A_C\cos^2\theta + A_N\right)\gamma^0, \\ \not{A}_{22} &= U^*_{e2}\not{B}U_{e2} + U^*_{\mu 2}\not{C}U_{\mu 2} = -\left(A_C\sin^2\theta + A_N\right)\gamma^0, \\ \not{A}_{12} &= U^*_{e1}\not{B}U_{e2} + U^*_{\mu 1}\not{C}U_{\mu 2} = -A_C\sin\theta\cos\theta\gamma^0, \\ \not{A}_{21} &= U^*_{e2}\not{B}U_{e1} + U^*_{\mu 2}\not{C}U_{\mu 1} = -A_C\sin\theta\cos\theta\gamma^0.\end{aligned} \tag{4.52}$$

4.2.2 Quantum Mechanical Treatment

A QM treatment of neutrino oscillations in matter can be found in many books, for example [18, 20, 21]. There is also a QM treatment of neutrinos propagating through matter using a wave packet approach [23].

The QM probability expression for neutrino oscillations in matter is

$$P_{e\mu}(L)_{\text{matter}} = \sin^2 2\theta^M \sin^2 x \tag{4.53}$$

where

$$\sin^2 2\theta^M = \frac{\sin^2 2\theta}{\sin^2 2\theta + \left(\cos 2\theta - \frac{2E}{\Delta m^2}A_C\right)^2}, \tag{4.54}$$

$$x = \frac{\Delta m^2 L}{4E}\sqrt{\sin^2 2\theta + \left(\cos 2\theta - \frac{2E}{\Delta m^2}A_C\right)^2}. \tag{4.55}$$

From [23] the QM probability for neutrino oscillations using a wave packet approach is

$$\begin{aligned}P_{\alpha\beta}(L)_{\text{matter}} &= \sum_{j,k} U_{\beta j}U^*_{\alpha j}U^*_{\beta k}U_{\alpha k} e^{-2\pi i \frac{L}{L^{\text{osc}}_{jk}(L)}} e^{-\left(\frac{L}{L^{\text{coh}}_{jk}(L)}\right)^2} \\ &\quad \times e^{-\frac{(E_j - E_k)^2}{8\sigma_p^2}},\end{aligned} \tag{4.56}$$

where the effective oscillation and coherence lengths are defined by

$$L_{jk}^{\rm osc}(L) \equiv \frac{4\pi E_0 L}{\int_0^L dx \Delta\mu_{jk}^2(x)}, \tag{4.57}$$

$$L_{jk}^{\rm coh}(L) \equiv \frac{2\sqrt{2}\sigma_x L}{\mid \int_0^L dx \Delta v_{jk}(x) \mid}. \tag{4.58}$$

Here $\Delta\mu_{jk}^2(x)$ is the effective mass squared difference in matter. In the relativistic limit, we have

$$\int_0^L dx \Delta v_{jk}(x) \approx \frac{1}{2E_0^2}(-1 + E_0\partial_{E_0}) \int_0^L dx \Delta\mu_{jk}^2(x). \tag{4.59}$$

In a static uniform background and for only two neutrino flavors, ν_e and ν_μ, $\Delta\mu_{jk}^2(x)$ takes on the form

$$\Delta\mu_{jk}^2(x) = \sqrt{\Delta_0^2 \sin^2 2\theta + (\Delta_0 \cos 2\theta - 2E_0 A_C)^2}, \tag{4.60}$$

where $\Delta_0 = m_2^2 - m_1^2$ is the mass squared difference in vacuum. Using this, gives

$$\int_0^L dx \Delta\mu_{jk}^2(x) = L\sqrt{\Delta_0^2 \sin^2 2\theta + (\Delta_0 \cos 2\theta - 2E_0 A_C)^2}. \tag{4.61}$$

Thus, the oscillation and coherence lengths are

$$L_{jk}^{\rm osc}(L) = \frac{4\pi E_0}{\sqrt{\Delta_0^2 \sin^2 2\theta + (\Delta_0 \cos 2\theta - 2E_0 A_C)^2}}, \tag{4.62}$$

$$L_{jk}^{\rm coh}(L) = \frac{4\sqrt{2}\sigma_x E_0^2 / \sqrt{\Delta_0^2 \sin^2 2\theta + (\Delta_0 \cos 2\theta - 2E_0 A_C)^2}}{\mid 1 + \frac{2E_0 A_C(\Delta_0 \cos 2\theta - 2E_0 A_C)}{\Delta_0^2 \sin^2 2\theta + (\Delta_0 \cos 2\theta - 2E_0 A_C)^2} \mid}. \tag{4.63}$$

4.2.3 Comments and Conclusions

We cannot really compare our result to the QM result, since as stated earlier we have only included the first non-trivial contribution in our approach. In order to be able to make a comparison with QM, we should either sum up all different contributions or using the complete propagator by solving the system of PDEs (4.17). We stopped at the expression (4.46), but one could always insert Eqs. (4.48) and (4.52), and try to simplify or rewrite the expression in a nice way in the case of two neutrino flavors. We are not going to follow this way here but instead make an interesting observation. Equation (4.46) contains the interesting factor $\frac{2|\mathbf{P}|A_{jk}}{m_j^2 - m_k^2}$. Since $A_{jk} \sim A_C$ and $|\mathbf{P}| \sim E$, we have, $\frac{2|\mathbf{P}|A_{jk}}{m_j^2 - m_k^2} \sim \frac{2EA_C}{m_j^2 - m_k^2}$. But this factor also arise in the QM treatment we therefore conclude that our approach to computed the transition probability for neutrinos propagating through matter using QFT should work. Of course, to obtain all the details, i.e., for example the resonance effect one sees in QM, one has to sum up all the different contributions.

Chapter 5

Comparing other Models with our QFT Treatment

5.1 Discussion of our Model

We have derived the following expressions for the oscillation probability between neutrinos with flavors α and β:

1. $\mathbf{P}_j = \mathbf{P}$ and $\mathbf{Q}_j = \mathbf{Q}$, which yields

$$\begin{aligned}\overline{P}^{rs}_{\alpha\beta}(\cdots) &= \sum_{j,k} U_{\beta j} U^*_{\alpha j} U^*_{\beta k} U_{\alpha k} \delta^{rs} \left(\frac{2\sigma_p \sigma_d}{\sigma_p^2 + \sigma_d^2}\right)^3 e^{-\frac{(\mathbf{P}-\mathbf{Q})^2}{2(\sigma_p^2+\sigma_d^2)}} \\ &\quad \times e^{-\frac{(m_j^2-m_k^2)^2(\sigma_p^2+\sigma_d^2)^3}{32|\mathbf{P}\sigma_d^2+\mathbf{Q}\sigma_p^2|^2\sigma_p^2\sigma_d^2}} e^{-\frac{i(m_j^2-m_k^2)(\sigma_p^2+\sigma_d^2)\mathbf{L}\cdot\hat{\mathbf{n}}}{2|\mathbf{P}\sigma_d^2+\mathbf{Q}\sigma_p^2|}}, \qquad (5.1)\end{aligned}$$

where $\hat{\mathbf{n}} = \frac{\mathbf{P}\sigma_d^2+\mathbf{Q}\sigma_p^2}{|\mathbf{P}\sigma_d^2+\mathbf{Q}\sigma_p^2|}$ and

2. $\mathbf{P}_j = \mathbf{P}$, $\mathbf{Q}_j = \mathbf{Q}$ and $\sigma_p = \sigma_d$, which yields

$$\begin{aligned}\overline{P}^{rs}_{\alpha\beta}(\cdots) &= \sum_{j,k} U_{\beta j} U^*_{\alpha j} U^*_{\beta k} U_{\alpha k} \delta^{rs} e^{-\frac{(\mathbf{P}-\mathbf{Q})^2}{4\sigma_p^2}} e^{-\frac{(m_j^2-m_k^2)^2}{4\sigma_p^2|\mathbf{P}+\mathbf{Q}|^2}} \\ &\quad \times e^{-\frac{i(m_j^2-m_k^2)\mathbf{L}\cdot\hat{\mathbf{n}}}{|\mathbf{P}+\mathbf{Q}|}} \qquad (5.2)\end{aligned}$$

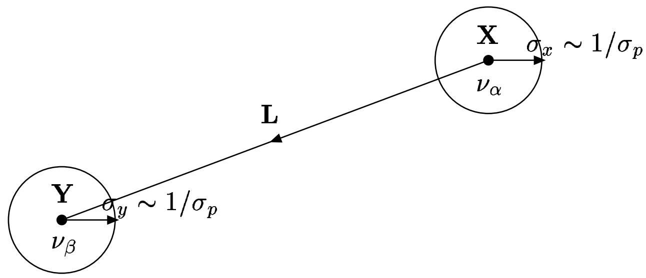
where $\hat{\mathbf{n}} = \frac{\mathbf{P}+\mathbf{Q}}{|\mathbf{P}+\mathbf{Q}|}$.

An important comment regarding δ^{rs} is necessary. Since we have assumed that the flavor α is created with spin r, the δ-function, δ^{rs}, ensures that the detected flavor β (which we assume to have spin s) has the same spin. This must be the case, because there is no processes which changes the spin, at least not with our assumptions.

5.1.1 Comments on the Different Exponential Factors

In the expressions for the probability (5.1) and (5.2), there are many different exponential factors, we are going to discuss them one by one, explaining their meaning. We will discuss case 2, since that expression is quite nice, not as involved as case 1, although the meaning is the same.

In case 2, we have assumed that a neutrino with flavor α is produced at $\mathbf{X}$ with momentum $\mathbf{P}$ and a corresponding uncertainty σ_p. At $\mathbf{Y}$ we detect a neutrino with flavor β which has momentum $\mathbf{Q}$ and uncertainty σ_p. This is visualized in the following figure.



First, the exponential factor

$$e^{-\frac{(\mathbf{P}-\mathbf{Q})^2}{4\sigma_p^2}} \tag{5.3}$$

is almost unity if the momentum difference $\mathbf{P}-\mathbf{Q}$ is smaller than the uncertainty σ_p; on the other hand, if $\mathbf{P}-\mathbf{Q}$ is much larger than σ_p, then it is almost zero. Thus, this factor ensures that the momentum difference between the produced and detected neutrinos must be much smaller than σ_p, or otherwise, there are no oscillations, since then $\overline{P}^{rs}_{\alpha\beta}(\cdots) \approx 0$. The case $\mathbf{P}=\mathbf{Q}$ in which the factor is 1 is the usual assumption one makes when deriving the oscillation formula.

Second, the exponential factor

$$e^{-\frac{(m_j^2-m_k^2)^2}{4\sigma_p^2|\mathbf{P}+\mathbf{Q}|^2}} \tag{5.4}$$

suppresses the interference if $m_j^2 - m_k^2 \gtrsim \sigma_p|\mathbf{P}+\mathbf{Q}|$. Since

$$\begin{aligned}\langle E_j\rangle - \langle E_k\rangle &\approx \left(|\langle\mathbf{p}\rangle_j| + \frac{m_j^2}{2|\langle\mathbf{p}\rangle_j|}\right) - \left(|\langle\mathbf{p}\rangle_k| + \frac{m_k^2}{2|\langle\mathbf{p}\rangle_k|}\right)\\ &= \{\mathbf{P}_j = \mathbf{P},\, \mathbf{Q}_j = \mathbf{Q},\, \sigma_p = \sigma_d \Rightarrow \langle\mathbf{p}\rangle_j = \langle\mathbf{p}\rangle_k = (\mathbf{P}+\mathbf{Q})/2\}\\ &= \frac{m_j^2 - m_k^2}{|\mathbf{P}+\mathbf{Q}|},\end{aligned} \tag{5.5}$$

we must have $\langle E_j\rangle - \langle E_k\rangle \lesssim \sigma_p$ in order to have interference. Thus, this factor which is due to the time integration ensures energy conservation within the uncertainty σ_p.

The last exponential factor is the phase factor

$$e^{-\frac{i(m_j^2-m_k^2)\mathbf{L}\cdot\hat{\mathbf{n}}}{|\mathbf{P}+\mathbf{Q}|}}, \tag{5.6}$$

which gives the neutrino oscillations as a function of the distance $\mathbf{L}\cdot\hat{\mathbf{n}} \approx |\mathbf{L}|$. The oscillation length is given by

$$L_{jk}^{\rm osc} = \frac{2\pi|\mathbf{P}+\mathbf{Q}|}{m_j^2 - m_k^2}. \tag{5.7}$$

5.2 Other Models

There are many different ways to arrive at an expression for the oscillation probability. We will here just state the final results from different approaches.

1. The simple standard QM expression is

$$P_{\alpha\beta}(x) = \sum_{j,k} U_{\beta j}U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} e^{-i\frac{(m_j^2-m_k^2)}{2|\mathbf{P}|}x}. \tag{5.8}$$

2. Using a QM wave packet formalism [9], one obtains

$$\begin{aligned} P_{\alpha\beta}(x) &= \left(\sum_{j'} \frac{|U_{\alpha j'}|^2}{|v_{j'}|}\right)^{-1} \sum_{j,k} U_{\beta j}U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} \left(\frac{2}{v_j^2+v_k^2}\right)^{1/2} \\ &\times e^{i\left[\langle p_j\rangle - \langle p_k\rangle - (\langle E_j\rangle - \langle E_k\rangle)\left[\frac{v_j+v_k}{v_j^2+v_k^2}\right]\right]x} \\ &\times e^{\left[-\frac{x^2}{2\sigma_x^2}\frac{(v_j-v_k)^2}{v_j^2+v_k^2} - \frac{(\langle E_j\rangle - \langle E_k\rangle)^2}{2\sigma_p^2(v_j^2+v_k^2)}\right]}. \end{aligned} \tag{5.9}$$

3. Blasone et al. [5] has derived an exact formula for neutrino oscillations between two flavors in QFT using a Green's function formalism. The probabilities are

$$\begin{aligned} P_{ee}(\mathbf{p},t) &= 1 - \sin^2 2\theta \left(|U(\mathbf{p})|^2 \sin^2 \frac{E_2(\mathbf{p}) - E_1(\mathbf{p})}{2} t \right. \\ &\left. + |V(\mathbf{p})|^2 \sin^2 \frac{E_2(\mathbf{p}) + E_1(\mathbf{p})}{2} t \right), \end{aligned} \tag{5.10}$$

$$P_{e\mu}(\mathbf{p},t) = 1 - P_{ee}(\mathbf{p},t), \tag{5.11}$$

where

$$\begin{aligned} |U(\mathbf{p})| &\equiv \sqrt{\frac{m_1 + E_1(\mathbf{p})}{2E_1(\mathbf{p})}}\sqrt{\frac{m_2 + E_2(\mathbf{p})}{2E_2(\mathbf{p})}} \\ &\times \left(1 + \frac{\mathbf{p}^2}{(m_1 + E_1(\mathbf{p}))(m_2 + E_2(\mathbf{p}))}\right), \qquad (5.12) \\ |V(\mathbf{p})| &\equiv |\mathbf{p}|\sqrt{\frac{m_1 + E_1(\mathbf{p})}{2E_1(\mathbf{p})}}\sqrt{\frac{m_2 + E_2(\mathbf{p})}{2E_2(\mathbf{p})}} \\ &\times \left(\frac{1}{m_2 + E_2(\mathbf{p})} - \frac{1}{m_1 + E_1(\mathbf{p})}\right), \qquad (5.13) \\ E_j(\mathbf{p}) &\equiv \sqrt{\mathbf{p}^2 + m_j^2}, \quad j = 1, 2. \qquad (5.14) \end{aligned}$$

The QM formulas (2.23) and (2.24) are obtained in the relativistic limit $|\mathbf{p}| \gg \sqrt{m_1 m_2}$, since $|U(\mathbf{p})| \to 1$ and $|V(\mathbf{p})| \to 0$. Thus, in the relativistic limit, Eqs. (5.10) and (5.11) are just another way of writing Eq. (5.8) in the case of two flavors.

We are also interested in what happens to the QM wave packet formalism (5.9) in the relativistic limit. The phase factor changes according to

$$e^{i\left[\langle p_j\rangle - \langle p_k\rangle - (\langle E_j\rangle - \langle E_k\rangle)\left[\frac{v_j+v_k}{v_j^2+v_k^2}\right]\right]x} \xrightarrow{\text{rel. lim.}} e^{-i\frac{(m_j^2-m_k^2)}{2|\mathbf{p}|}x}. \qquad (5.15)$$

One interesting thing about Eq. (5.9) is the exponential factor

$$e^{\left[-\frac{x^2}{2\sigma_x^2}\frac{(v_j-v_k)^2}{v_j^2+v_k^2}\right]}. \qquad (5.16)$$

The mass eigenstate wave packets propagate with different velocities, thus, they overlap and interfere only for a finite distance. This implies the existence of a coherence length for neutrino oscillations, this is one of the predictions of the wave packet treatment. From Eq. (5.16) one obtains the coherence length

$$L_{jk}^{\text{coh}} = \sqrt{2}\sigma_x \left[\frac{v_j^2 + v_k^2}{(v_j - v_k)^2}\right]^{1/2}, \quad j \neq k. \qquad (5.17)$$

In the relativistic limit, one finds,

$$L_{jk}^{\text{coh}} = \sqrt{2}\sigma_x \left[\frac{v_j^2 + v_k^2}{(v_j - v_k)^2}\right]^{1/2} \xrightarrow{\text{rel. lim.}} L_{jk}^{\text{coh}} \simeq \frac{2\sigma_x}{|v_j - v_k|}. \qquad (5.18)$$

which can be simplified to

$$L_{jk}^{\text{coh}} \simeq \frac{4\sigma_x \langle p\rangle^2}{m_j^2 - m_k^2}, \qquad (5.19)$$

using $\langle E_j \rangle = \sqrt{\langle p_j \rangle^2 + m_j^2}$ and $v_j = \frac{\langle p_j \rangle}{\langle E_j \rangle}$. Inserting Eqs. (5.15), (5.16), and (5.19) into Eq. (5.9) we obtain

$$P_{\alpha\beta}^{\text{rel. lim.}}(x) = \left(\sum_{j'} \frac{|U_{\alpha j'}|^2}{|v_{j'}|} \right)^{-1} \sum_{j,k} U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} \times e^{-i\frac{(m_j^2 - m_k^2)}{2\langle p \rangle} x} e^{-\frac{x^2 (m_j^2 - m_k^2)^2}{16\sigma_x^2 \langle p \rangle^4}} e^{-\frac{(m_j^2 - m_k^2)^2}{16\sigma_p^2 \langle p \rangle^2}}. \quad (5.20)$$

5.3 Comparison

Since we have derived our expressions (5.1) and (5.2) in the relativistic limit, it is only interesting to compare them to the QM wave packet formalism (5.20). There is no reason why we should compare them to the simple expression (5.8), since this expression does not contain all the interesting physics. Because Eq. (5.20) is one dimensional, we use the one dimensional analog of Eq. (5.2), which is

$$\overline{P}_{\alpha\beta}^{rs}(\cdots) = \sum_{j,k} U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} \delta^{rs} e^{-\frac{(P-Q)^2}{4\sigma_p^2}} e^{-\frac{(m_j^2 - m_k^2)^2}{4\sigma_p^2 |P+Q|^2}} \times e^{-\frac{i(m_j^2 - m_k^2)x}{|P+Q|}}. \quad (5.21)$$

We observe that there are similarities, but also a crucial difference between Eqs. (5.20) and (5.21). Both expressions contain the "same" oscillation and energy conservation terms, i.e.,

$$e^{-i\frac{(m_j^2 - m_k^2)}{2\langle p \rangle} x} \approx e^{-\frac{i(m_j^2 - m_k^2)x}{|P+Q|}}, \quad (5.22)$$

$$e^{-\frac{(m_j^2 - m_k^2)^2}{16\sigma_p^2 \langle p \rangle^2}} \approx e^{-\frac{(m_j^2 - m_k^2)^2}{4\sigma_p^2 |P+Q|^2}}, \quad (5.23)$$

since $P \approx Q$ beacuse of the exponential factor $e^{-\frac{(P-Q)^2}{4\sigma_p^2}}$. The important and crucial difference is that our expression (5.21) does not contain a damping factor $e^{-\frac{x^2 (m_j^2 - m_k^2)^2}{16\sigma_x^2 \langle p \rangle^4}}$, which defines the coherence length. A motivation for why we do not have such a damping factor is: We started with expression (3.78), the probability in the relativistic limit, and then studied some special cases, both with $\mathbf{P}_j = \mathbf{P}$ and $\mathbf{Q}_j = \mathbf{Q}$. In the relativistic limit, we made the approximation

$$\mathbf{v}_j = \frac{\langle \mathbf{p} \rangle_j}{\langle E_j \rangle} \approx \frac{\langle \mathbf{p} \rangle_j}{|\langle \mathbf{p} \rangle_j|}. \quad (5.24)$$

This approximation and the special case, $\mathbf{P}_j = \mathbf{P}$ and $\mathbf{Q}_j = \mathbf{Q}$, corresponds to using $v_j = v_k$ in Eq. (5.9), in which case the exponential factor $e^{\left[-\frac{x^2}{2\sigma_x^2} \frac{(v_j - v_k)^2}{v_j^2 + v_k^2} \right]} = 0$. There is therefore no damping factor in this approximation.

Our final conclusion is that our QFT treatment is similar to a QM treatment using wave packets in the relativistic limit.

Appendix A

Expressing a_j^r and b_j^r in terms of the Field $\nu_j(x)$

To express the operators $a_j^r(\mathbf{q})$ and $b_j^r(-\mathbf{q})$ in terms of the fields $\nu_j(x)$ and $\overline{\nu}_j(x)$ we proceed as follows:

$$
\begin{aligned}
&\int \frac{d^3\mathbf{x}}{(2\pi)^3} e^{iqx} \nu_j(x) \\
&= \int \frac{d^3\mathbf{x}}{(2\pi)^3} e^{iqx} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_j(\mathbf{p})}} \sum_s \left(a_j^s(\mathbf{p}) u_j^s(p) e^{-ipx} + b_j^{s\dagger}(\mathbf{p}) v_j^s(p) e^{ipx} \right) \\
&= \left\{ \int \frac{d^3\mathbf{x}}{(2\pi)^3} e^{i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} = \delta^{(3)}(\mathbf{q}-\mathbf{p}) \right\} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_j(\mathbf{p})}} \sum_s \Big(a_j^s(\mathbf{p}) u_j^s(p) e^{ix^0(q^0-p^0)} \delta^{(3)}(\mathbf{q}-\mathbf{p}) \\
&+ b_j^{s\dagger}(\mathbf{p}) v_j^s(p) e^{ix^0(q^0+p^0)} \delta^{(3)}(\mathbf{q}+\mathbf{p}) \Big) \\
&= \{ p^0 = E_j(\mathbf{p}) \text{ and } q^0 = E_j(\mathbf{q}) \} \\
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2E_j(\mathbf{q})}} \sum_s \left(a_j^s(\mathbf{q}) u_j^s(q) + b_j^{s\dagger}(-\mathbf{q}) v_j^s(q^0, -\mathbf{q}) e^{ix^0 2E_j(\mathbf{q})} \right) . \qquad \text{(A.1)}
\end{aligned}
$$

This gives

$$
\begin{aligned}
& u_j^{r\dagger}(q) \int \frac{d^3\mathbf{x}}{(2\pi)^3} e^{iqx} \nu_j(x) \\
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2E_j(\mathbf{q})}} \sum_s \Big(a_j^s(\mathbf{q}) u_j^{r\dagger}(q) u_j^s(q) \\
&+ b_j^{s\dagger}(-\mathbf{q}) u_j^{r\dagger}(q) v_j^s(q^0, -\mathbf{q}) e^{ix^0 2E_j(\mathbf{q})} \Big) \\
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2E_j(\mathbf{q})}} \sum_s \Big(a_j^s(\mathbf{q}) 2E_j(\mathbf{q}) \delta^{rs} + b_j^{s\dagger}(-\mathbf{q}) 0 e^{ix^0 2E_j(\mathbf{q})} \Big) \\
&= \frac{\sqrt{2E_j(\mathbf{q})}}{(2\pi)^3} a_j^r(\mathbf{q}). \qquad \text{(A.2)}
\end{aligned}
$$

We thus have

$$a_j^r(\mathbf{q}) = \frac{1}{\sqrt{2E_j(\mathbf{q})}} u_j^{r\dagger}(q) \int d^3\mathbf{x} e^{iqx} \nu_j(x). \qquad \text{(A.3)}$$

Similarly one obtains

$$b_j^r(\mathbf{q}) = \frac{1}{\sqrt{2E_j(\mathbf{q})}} v_j^{r\dagger}(q) \int d^3\mathbf{x} e^{-iqx} \nu_j(x). \qquad \text{(A.4)}$$

This expressions can also be found in [16] on page 147.

Appendix B

Calculation of the Propagator in Vacuum

We are interested in simplifying the following expression as much as possible:

$$\int d^3\mathbf{y}' e^{iqy'} \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + m_j)}{k^2 - m_j^2 + i\epsilon} e^{-ik(y'-x')} \int d^3\mathbf{x}' e^{-ipx'}. \tag{B.1}$$

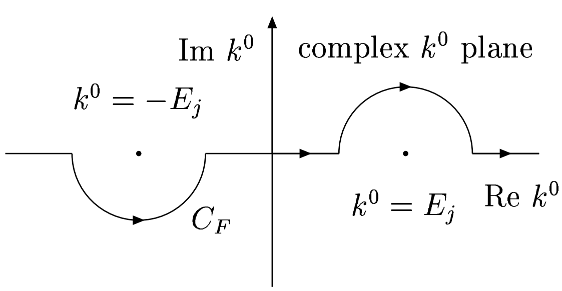
The Feynman fermion propagator is defined by

$$S_j(y' - x') = \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + m_j)}{k^2 - m_j^2 + i\epsilon} e^{-ik(y'-x')}. \tag{B.2}$$

The integration is to be performed in the following way

$$\lim_{\epsilon \to 0} \int \frac{d^3\mathbf{k} dk^0}{(2\pi)^4} \frac{i(\not{k} + m_j)}{k^2 - m_j^2 + i\epsilon} e^{-ik(y'-x')} = \int_{C_F} \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + m_j)}{k^2 - m_j^2} e^{-ik(y'-x')}, \tag{B.3}$$

where the integration in the complex k^0 plane is along the whole real axis $-\infty < k^0 < \infty$ and the contour C_F is defined according to the figure.



This corresponds to moving the poles $k^0 = \pm E$ according to

$$k^0 = \pm E \to k^0 = \pm(E - i\epsilon). \tag{B.4}$$

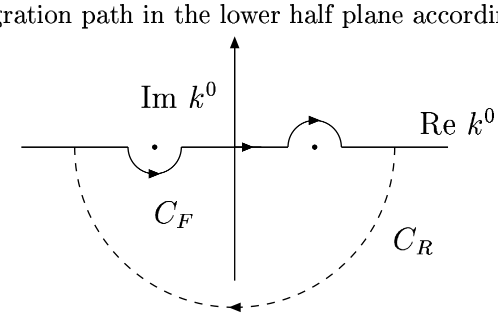
A straightforward calcuation gives

$$\begin{aligned}
&\int d^3\mathbf{y}' e^{iqy'} \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + m_j)}{k^2 - m_j^2 + i\epsilon} e^{-ik(y'-x')} \int d^3\mathbf{x}' e^{-ipx'} \\
&= \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + m_j)}{k^2 - m_j^2 + i\epsilon} \int d^3\mathbf{x}' e^{-ix'(p-k)} \int d^3\mathbf{y}' e^{-iy'(k-q)} \\
&= \left\{ \int d^3\mathbf{x}' e^{-ix'(p-k)} = \int d^3\mathbf{x}' e^{-ix'^0(p^0-k^0)} e^{i\mathbf{x}'\cdot(\mathbf{p}-\mathbf{k})} \right. \\
&= \left. (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{k}) e^{-ix'^0(p^0-k^0)} \right\} \\
&= \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + m_j)}{k^2 - m_j^2 + i\epsilon} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{k}) e^{-ix'^0(p^0-k^0)} \\
&\times (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{q}) e^{-iy'^0(k^0-q^0)} \\
&= (2\pi)^6 \delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{-ix'^0 p^0} e^{iy'^0 q^0} \\
&\times \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + m_j)}{k^2 - m_j^2 + i\epsilon} e^{-ik^0(y'^0-x'^0)} \delta^{(3)}(\mathbf{p}-\mathbf{k}) \\
&= (2\pi)^6 \delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{-ix'^0 p^0} e^{iy'^0 q^0} \\
&\times \int \frac{dk^0}{(2\pi)^4} \frac{i(\gamma^0 k_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3 + m_j)}{k_0^2 - p_1^2 - p_2^2 - p_3^2 - m_j^2 + i\epsilon} e^{-ik^0(y'^0-x'^0)}.
\end{aligned} \tag{B.5}$$

The problem is to compute the last integral. Since $y'^0 - x'^0 > 0$ in our problem we can compute the last integral in the following way

$$\begin{aligned}
&\int \frac{dk^0}{(2\pi)^4} \frac{i(\gamma^0 k_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3 + m_j)}{k_0^2 - p_1^2 - p_2^2 - p_3^2 - m_j^2 + i\epsilon} e^{-ik^0(y'^0-x'^0)} \\
&= \int_{C_F} \frac{dk^0}{(2\pi)^4} \frac{i(\gamma^0 k_0 + \mathbf{p}\cdot\boldsymbol{\gamma} + m_j)}{k_0^2 - E_j^2(\mathbf{p})} e^{-ik^0(y'^0-x'^0)}.
\end{aligned} \tag{B.6}$$

By closing the integration path in the lower half plane according to the figure



we have

$$
\begin{aligned}
&\int_{C_F} \frac{dk^0}{(2\pi)^4} \frac{i(\gamma^0 k_0 + \mathbf{p}\cdot\boldsymbol{\gamma} + m_j)}{k_0^2 - E_j^2(\mathbf{p})} e^{-ik^0(y'^0 - x'^0)} \\
&= \lim_{R\to\infty} \int_{C_F + C_R} \frac{dk^0}{(2\pi)^4} \frac{i(\gamma^0 k_0 + \mathbf{p}\cdot\boldsymbol{\gamma} + m_j)}{k_0^2 - E_j^2(\mathbf{p})} e^{-ik^0(y'^0 - x'^0)} \\
&= 2\pi i \frac{i}{(2\pi)^4} \frac{\gamma^0 E_j(\mathbf{p}) + \mathbf{p}\cdot\boldsymbol{\gamma} + m_j}{2E_j(\mathbf{p})} e^{-iE_j(\mathbf{p})(y'^0 - x'^0)} \\
&= -\frac{1}{(2\pi)^3} \frac{\not{p} + m_j}{2E_j(\mathbf{p})} e^{-iE_j(\mathbf{p})(y'^0 - x'^0)}.
\end{aligned} \tag{B.7}
$$

Where we have used Cauchy's residue theorem. Thus we finally have

$$
\begin{aligned}
&\int d^3\mathbf{y}' e^{iqy'} \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + m_j)}{k^2 - m_j^2 + i\epsilon} e^{-ik(y'-x')} \int d^3\mathbf{x}' e^{-ipx'} \\
&= -(2\pi)^2 \delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ix'^0 p^0} e^{iy'^0 q^0} \left(\not{p} + m_j\right) \frac{\pi e^{-i(y'^0 - x'^0)E_j(\mathbf{p})}}{E_j(\mathbf{p})}.
\end{aligned} \tag{B.8}
$$

Appendix C

Calculation of the Propagator in Matter

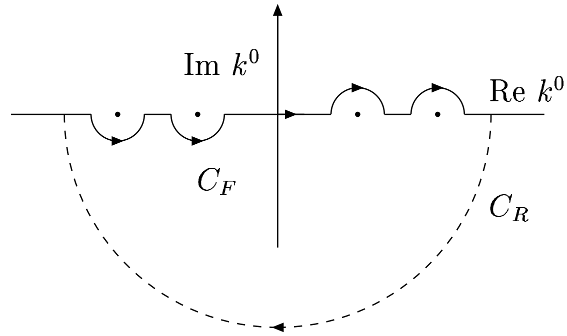
We are interested in simplifying the following expression as much as possible:

$$\int d^3\mathbf{y}' e^{iqy'} \int_{C_F} \frac{d^4k}{(2\pi)^4} \frac{1}{\not{k} - m_j} \not{A}_{jk} \frac{1}{\not{k} - m_k} e^{-ik(y'-x')} \int d^3\mathbf{x}' e^{-ipx'} \qquad \text{(C.1)}$$

where the contour C_F is defined according to the figure below. A straightforward calculation gives

$$\begin{aligned}
&\int d^3\mathbf{y}' e^{iqy'} \int_{C_F} \frac{d^4k}{(2\pi)^4} \frac{1}{\not{k} - m_j} \not{A}_{jk} \frac{1}{\not{k} - m_k} e^{-ik(y'-x')} \int d^3\mathbf{x}' e^{-ipx'} \\
&= \{\text{Same steps as in Eq. (B.5)}\} \\
&= (2\pi)^6 \delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ix'^0 p^0} e^{iy'^0 q^0} \\
&\times \int_{C_F} \frac{d^4k}{(2\pi)^4} \frac{1}{\not{k} - m_j} \not{A}_{jk} \frac{1}{\not{k} - m_k} e^{-ik^0(y'^0 - x'^0)} \delta^{(3)}(\mathbf{p} - \mathbf{k}) \\
&= (2\pi)^6 \delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ix'^0 p^0} e^{iy'^0 q^0} \\
&\times \int_{C_F} \frac{d^4k}{(2\pi)^4} \frac{\not{k} + m_j}{k^2 - m_j^2} \not{A}_{jk} \frac{\not{k} + m_k}{k^2 - m_k^2} e^{-ik^0(y'^0 - x'^0)} \delta^{(3)}(\mathbf{p} - \mathbf{k}) \\
&= (2\pi)^6 \delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ix'^0 p^0} e^{iy'^0 q^0} \\
&\times \int_{C_F} \frac{dk_0}{(2\pi)^4} \frac{\left(\gamma^0 k_0 + \mathbf{p} \cdot \boldsymbol{\gamma} + m_j\right) \not{A}_{jk} \left(\gamma^0 k_0 + \mathbf{p} \cdot \boldsymbol{\gamma} + m_k\right)}{\left(k_0^2 - E_j^2(\mathbf{p})\right)\left(k_0^2 - E_k^2(\mathbf{p})\right)} \\
&\times e^{-ik^0(y'^0 - x'^0)},
\end{aligned} \qquad \text{(C.2)}$$

where $p_0 = E_k(\mathbf{p})$ and $q_0 = E_j(\mathbf{q})$. By closing the integration path in the lower half plane (this can be done since $y'^0 - x'^0 > 0$) according to the figure below



we have

$$\begin{aligned}
&\int d^3\mathbf{y}' e^{iqy'} \int_{C_F} \frac{d^4k}{(2\pi)^4} \frac{1}{\not{k} - m_j} \not{A}_{jk} \frac{1}{\not{k} - m_k} e^{-ik(y'-x')} \int d^3\mathbf{x}' e^{-ipx'} \\
&= (2\pi)^6 \delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ix'^0 E_k(\mathbf{p})} e^{iy'^0 E_j(\mathbf{q})} \\
&\times \lim_{R\to\infty} \int_{C_F + C_R} \frac{dk_0}{(2\pi)^4} \frac{\left(\gamma^0 k_0 + \mathbf{p}\cdot\boldsymbol{\gamma} + m_j\right) \not{A}_{jk} \left(\gamma^0 k_0 + \mathbf{p}\cdot\boldsymbol{\gamma} + m_k\right)}{\left(k_0^2 - E_j^2(\mathbf{p})\right)\left(k_0^2 - E_k^2(\mathbf{p})\right)} \\
&\times e^{-ik^0(y'^0 - x'^0)} \\
&= (2\pi)^2 \delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ix'^0 E_k(\mathbf{p})} e^{iy'^0 E_j(\mathbf{q})} 2\pi i \\
&\times \left[\left(\frac{(\not{p}_j + m_j) \not{A}_{jk} (\not{p}_j + m_k)}{2E_j(\mathbf{p}) \left(E_j^2(\mathbf{p}) - E_k^2(\mathbf{p})\right)} \right) e^{-iE_j(\mathbf{p})(y'^0 - x'^0)} \right. \\
&\left. + \left(\frac{(\not{p}_k + m_j) \not{A}_{jk} (\not{p}_k + m_k)}{2E_k(\mathbf{p}) \left(E_k^2(\mathbf{p}) - E_j^2(\mathbf{p})\right)} \right) e^{-iE_k(\mathbf{p})(y'^0 - x'^0)} \right].
\end{aligned} \tag{C.3}$$

Here we have used Cauchy's residue theorem and introduced the notation $\not{p}_j \equiv \gamma^0 E_j(\mathbf{p}) + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3$.

Appendix D

Normalization

In order for the state $|\psi_\alpha(\mathbf{X},\mathbf{P},T)\rangle$ to be properly normalized, we require that $\langle\psi_\alpha(\mathbf{X},\mathbf{P},T)|\psi_\alpha(\mathbf{X},\mathbf{P},T)\rangle = 1$. This will give us a condition which the distribution function $F(\mathbf{X},\mathbf{P},T,\mathbf{p})$ must satisfy. Since

$$|\psi_\alpha(\mathbf{X},\mathbf{P},T)\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\alpha(\mathbf{p})}}F(\mathbf{X},\mathbf{P},T,\mathbf{p})|\psi_\alpha(\mathbf{p})\rangle, \tag{D.1}$$

we find that (use the short notation $\langle\cdot|\cdot\rangle = \langle\psi_\alpha(\mathbf{X},\mathbf{P},T)|\psi_\alpha(\mathbf{X},\mathbf{P},T)\rangle$)

$$\begin{aligned}
\langle\cdot|\cdot\rangle &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{F(\mathbf{X},\mathbf{P},T,\mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{F^\dagger(\mathbf{X},\mathbf{P},T,\mathbf{q})}{\sqrt{2E_\alpha(\mathbf{q})}}\langle\psi_\alpha(\mathbf{q})|\psi_\alpha(\mathbf{p})\rangle \\
&= \left\{\langle\psi_\alpha(\mathbf{q})|\psi_\alpha(\mathbf{p})\rangle = \sum_{j,k} U_{\alpha j}U^*_{\alpha k}\langle\nu_j(\mathbf{q})|\nu_k(\mathbf{p})\rangle\right. \\
&= \sum_{j,k} U_{\alpha j}U^*_{\alpha k}2E_j(\mathbf{p})(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{jk} \\
&= \left.\sum_j U_{\alpha j}U^*_{\alpha j}2E_j(\mathbf{p})(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\right\} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{F(\mathbf{X},\mathbf{P},T,\mathbf{p})}{\sqrt{2E_\alpha(\mathbf{p})}}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{F^\dagger(\mathbf{X},\mathbf{P},T,\mathbf{q})}{\sqrt{2E_\alpha(\mathbf{q})}} \\
&\quad\times\sum_j U_{\alpha j}U^*_{\alpha j}2E_j(\mathbf{p})(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{F(\mathbf{X},\mathbf{P},T,\mathbf{p})F^\dagger(\mathbf{X},\mathbf{P},T,\mathbf{p})}{2E_\alpha(\mathbf{p})}\sum_j U_{\alpha j}U^*_{\alpha j}2E_j(\mathbf{p}).
\end{aligned} \tag{D.2}$$

We now define $E_\alpha(\mathbf{p}) \equiv \sum_j U_{\alpha j} U^*_{\alpha j} E_j(\mathbf{p})$, this gives

$$\langle\cdot|\cdot\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3} |F(\mathbf{X}, \mathbf{P}, T, \mathbf{p})|^2. \tag{D.3}$$

We thus have that the distribution function $F(\mathbf{X}, \mathbf{P}, T, \mathbf{p})$ must satisfy

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} |F(\mathbf{X}, \mathbf{P}, T, \mathbf{p})|^2 = 1. \tag{D.4}$$

Finally, we are going to check that this condition is fulfilled for our Gaussian expression

$$F(\mathbf{X}, \mathbf{P}, T, \mathbf{p}) = \left(\frac{2\pi}{\sigma_p^2}\right)^{3/4} e^{-\frac{(\mathbf{p}-\mathbf{P})^2}{4\sigma_p^2}} e^{-i(\mathbf{p}\cdot\mathbf{X} - E(\mathbf{p})T)}. \tag{D.5}$$

We obtain

$$\begin{aligned}
\int \frac{d^3\mathbf{p}}{(2\pi)^3} |F(\mathbf{X}, \mathbf{P}, T, \mathbf{p})|^2 &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{2\pi}{\sigma_p^2}\right)^{3/2} e^{-\frac{(\mathbf{p}-\mathbf{P})^2}{4\sigma_p^2} - \frac{(\mathbf{p}-\mathbf{P})^2}{4\sigma_p^2}} \\
&\quad \times e^{i\mathbf{X}\cdot(\mathbf{p}-\mathbf{p})} e^{iT(E(\mathbf{p})-E(\mathbf{p}))} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{2\pi}{\sigma_p^2}\right)^{3/2} e^{-\frac{(\mathbf{p}-\mathbf{P})^2}{2\sigma_p^2}} \\
&= \left\{ \int d^3\mathbf{p} e^{-\frac{(\mathbf{p}-\mathbf{P})^2}{2\sigma_p^2}} = (2\pi)^{3/2}\sigma_p^3 \right\} = 1.
\end{aligned} \tag{D.6}$$

Thus, our $F(\mathbf{X}, \mathbf{P}, T, \mathbf{p})$ is properly normalized.

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