B

The dispersion and Fresnel coefficients for a bianisotropic medium

We will present in this appendix, principally for the benefit of the less experienced reader, a derivation of the dispersion in a bianisotropic medium and the Fresnel coefficients across the interface between an isotropic medium and a bianisotropic medium. This is primarily to only demonstrate the manner of calculating them. By following the procedure the reader should be able to derive all the cases (anisotropic, bi-isotropic or bianisotropic) that (s)he would confront in the book. We will consider the case when the constitutive relations for the bianisotropic medium are given by

$$\mathbf{D} = \varepsilon_0 \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \mathbf{E} + \frac{1}{c} \begin{pmatrix} 0 & \mathrm{i}\xi_{xy} & 0 \\ 0 & 0 & 0 \\ 0 & \mathrm{i}\xi_{zy} & 0 \end{pmatrix} \mathbf{H}, \tag{B.1}$$

$$\mathbf{B} = \frac{1}{c} \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{i}\zeta_{yx} & 0 & \mathrm{i}\zeta_{yz} \\ 0 & 0 & 0 \end{pmatrix} \mathbf{E} + \mu_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{H}. \tag{B.2}$$

These relations are seen to be reciprocal if $\zeta_{yx} = -\xi_{xy}$ and $\zeta_{yz} = -\xi_{zy}$. An array of split-ring resonators with their axes along the y direction (or cylinders along the y directions) would be well described by these constitutive relations (see Chapter 3). This is a special case of a bianisotropic medium where the modes in the medium are linearly polarized.

We will consider the plane of incidence to be the x-z plane (zero k_y component) and the case of TM modes (The magnetic field, H_y, is normal to the plane of incidence). Hence we have

$$H_x = H_z = E_y = 0. \tag{B.3}$$

The constituent relations can be rewritten in their components:

$$D_x = \varepsilon_0 \varepsilon_x E_x + \mathrm{i}\frac{\xi_{xy}}{c} H_y, \tag{B.4}$$

$$D_z = \varepsilon_0 \varepsilon_z E_z + \mathrm{i}\frac{\xi_{zy}}{c} H_y, \tag{B.5}$$

$$B_y = \mu_0 \mu_y H_y - \mathrm{i}\frac{\xi_{xy}}{c} E_x - \mathrm{i}\frac{\xi_{zy}}{c} E_z. \tag{B.6}$$

Now consider the Maxwell equations for plane waves

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}, \qquad \mathbf{k} \times \mathbf{H} = -\omega \mathbf{D}. \tag{B.7}$$

Assuming a complete y-invariance and since we have $k_y = 0$, we can expand these equations in the bianisotropic medium as

$$k_z E_x - k_x E_z = +\omega B_y = \omega[\mu_0 \mu_y H_y - \mathrm{i}\frac{\xi_{xy}}{c} E_x - \mathrm{i}\frac{\xi_{zy}}{c} E_z], \tag{B.8}$$

$$-k_z H_y = -\omega D_x = -\omega[\varepsilon_0 \varepsilon_x E_x + \mathrm{i}\frac{\xi_{xy}}{c} H_y], \tag{B.9}$$

$$k_x H_y = -\omega D_z = -\omega[\varepsilon_0 \varepsilon_z E_z + \mathrm{i}\frac{\xi_{zy}}{c} H_y], \tag{B.10}$$

from which we obtain

$$E_z = \frac{1}{-\varepsilon_0 \varepsilon_z}[\frac{k_x}{\omega} + \mathrm{i}\frac{\xi_{zy}}{c}] H_y, \tag{B.11}$$

$$E_x = \frac{1}{\varepsilon_0 \varepsilon_x}(\frac{k_z}{\omega} - \mathrm{i}\frac{\xi_{xy}}{c}) H_y, \tag{B.12}$$

$$(\frac{k_z}{\omega} + i\frac{\xi_{xy}}{c}) E_x = \left[\mu_0 \mu_y - \frac{1}{\varepsilon_0 \varepsilon_z}(\frac{k_x^2}{\omega^2} + \frac{\xi_{zy}^2}{c^2})\right] H_y. \tag{B.13}$$

For a non-trivial solution, we will require that the determinant for the last two homogeneous equations (for E_x and H_y) vanishes. This gives us the dispersion of the waves in this medium:

$$k_z^2 = \left(\varepsilon_x \mu_y - \xi_{xy}^2 - \frac{\varepsilon_x}{\varepsilon_z}\xi_{zy}^2\right)\frac{\omega^2}{c^2} - \frac{\varepsilon_x}{\varepsilon_z} k_x^2, \tag{B.14}$$

or to put it more symmetrically

$$\varepsilon_x k_x^2 + \varepsilon_z k_z^2 + \varepsilon_z \xi_{xy}^2 \omega^2/c^2 + \varepsilon_x \xi_{zy}^2 \omega^2/c^2 = \varepsilon_x \varepsilon_z \mu_y \omega^2/c^2. \tag{B.15}$$

Here we can see that $n_{\text{eff}} = \pm\sqrt{\varepsilon_x \mu_y - \xi_{xy}^2 - \xi_{zy}^2}$ acts like the refractive index when $\varepsilon_x = \varepsilon_z$. The sign of the square root will be determined by whether the real part of the quantity inside the square root is positive or negative (see Chapter 5 for details).

Now we will calculate the Fresnel coefficients for the reflected and transmitted wave amplitudes. Let us consider the reflection or transmission of a plane wave (moving along $+z$ direction) incident from vacuum onto the bianisotropic medium. Since we consider here P-polarised light, we can write for the incident, reflected and the transmitted fields, and the corresponding wave-vectors:

$$H_{iy} = e^{\mathrm{i}(k_x x + k_{1z} z)}, \tag{B.16}$$

$$H_{ry} = r_{\mathrm{mp}}\ e^{\mathrm{i}(k_x x - k_{1z} z)}, \tag{B.17}$$

$$H_{ty} = t_{\mathrm{pp}}\ e^{\mathrm{i}(k_x x + k_{2z} z)}, \tag{B.18}$$

$$k_{1z}^2 = \frac{\omega^2}{c^2} - k_x^2, \tag{B.19}$$

$$k_{2z}^2 = [\varepsilon_x \mu_y - \xi_{xy}^2 - \xi_{zy}^2]\frac{\omega^2}{c^2} - k_x^2. \tag{B.20}$$

The requirement of continuity of the parallel component of $\mathbf{H} \ \Rightarrow k_x = \mathrm{const}$ across the interface and that

$$1 + r_{\mathrm{mp}} = t_{\mathrm{pp}}, \tag{B.21}$$

where the subscripts "mp" imply reflection of a positive-going wave into the negative direction and the subscripts "pp" imply the transmission into the positive direction of a wave initially moving in the positive direction. Noting that

$$E_{2x} = \frac{1}{\varepsilon_0 \varepsilon_x}(\frac{k_{2z}}{\omega} - \mathrm{i}\frac{\xi_{xy}}{c})H_{2y}, \tag{B.22}$$

$$E_{1x} = \frac{k_{1z}}{\varepsilon_0 \varepsilon_1 \omega} H_{1y} \tag{B.23}$$

in the bianisotropic and isotropic medium respectively, the requirement of continuity of the parallel components of the electric field (E_x) yields

$$\frac{k_{z1}}{\varepsilon_0 \varepsilon_1 \omega} + \frac{-k_{z1}}{\varepsilon_0 \varepsilon_1 \omega} r_{\mathrm{mp}} = \frac{1}{\varepsilon_0 \varepsilon_x}(\frac{k_{2z}}{\omega} - \mathrm{i}\frac{\xi_{xy}}{c}) t_{\mathrm{pp}}. \tag{B.24}$$

Hence, we have for the reflection coefficient

$$r_{\mathrm{mp}} = \frac{k_{z1}/\varepsilon_1 - (k_{z2}/\varepsilon_1 - \mathrm{i}\xi_{xy}\omega/c)/\varepsilon_x}{k_{z1} + (k_{z2} - \mathrm{i}\xi_{xy}\omega/c)/\varepsilon_x}. \tag{B.25}$$

Similarly, the transmission coefficient across the interface is obtained as

$$t_{\mathrm{pp}} = \frac{2k_{z1}/\varepsilon_1}{k_{z1}/\varepsilon_1 + (k_{z2} - \mathrm{i}\xi_{xy}\omega/c)/\varepsilon_x}. \tag{B.26}$$

Next, let us calculate the Fresnel coefficients for a wave incident on an isotropic dielectric medium with ($\varepsilon_1 \mu_1$) from the bianisotropic medium. The bianisotropic medium is now assumed to occupy the negative half-space ($z <$ 0 and the vacuum the positive half-space ($z > 0$). Once again, using the continuity of the tangential components of the magnetic and electric fields, we have

$$1 + r'_{\mathrm{mp}} = t'_{\mathrm{pp}}, \tag{B.27}$$

$$\frac{1}{\varepsilon_0 \varepsilon_x}(\frac{k_{z2}}{\omega} - \mathrm{i}\frac{\xi_{xy}}{c}) + \frac{1}{\varepsilon_0 \varepsilon_x}(-\frac{k_{z2}}{\omega} - \mathrm{i}\frac{\xi_{xy}}{c}) r'_{\mathrm{mp}} = \frac{k_{z1}}{\varepsilon_0 \varepsilon_1 \omega} t'_{\mathrm{pp}}, \tag{B.28}$$

from which we obtain

$$r'_{\mathrm{mp}} = \frac{(k_{z2} - \mathrm{i}\ \xi_{xy}\omega/c)/\varepsilon_x - k_{z1}/\varepsilon_1}{(k_{z2} + \mathrm{i}\ \xi_{xy}\omega/c)/\varepsilon_x + k_{z1}/\varepsilon_1}, \tag{B.29}$$

$$t'_{\mathrm{pp}} = \frac{2k_{z2}/\varepsilon_x}{(k_{z2} + \mathrm{i}\ \xi_{xy}\omega/c)/\varepsilon_x + k_{z1}/\varepsilon_1} \tag{B.30}$$

We do not simply have $r_{\mathrm{mp}} = r'_{\mathrm{mp}}$ as in the case of an isotropic medium. In fact, we also do not have the same Fresnel coefficients if the bianisotropic medium occupied the positive half-space ($z > 0$), the isotropic medium occupied the negative-half space, and the wave were incident in the negative direction. This stems about from the fact that the impedance in a bianisotropic medium depends on the direction of propagation. We can call

$$\eta_p = \frac{\sqrt{\varepsilon_x\mu_y - \xi_{xy}^2 - \xi_{zy}^2 - (k_x/k_0)^2} - \mathrm{i}\xi_{xy}}{\varepsilon_x}, \tag{B.31}$$

$$\eta_m = \frac{-\sqrt{\varepsilon_x\mu_y - \xi_{xy}^2 - \xi_{zy}^2 - (k_x/k_0)^2} - \mathrm{i}\xi_{xy}}{\varepsilon_x} \tag{B.32}$$

as the effective impedances for the positive- and negative-going waves in the bianisotropic medium (where $k_0 = \omega/c$). Also defining

$$\eta_i = \frac{\sqrt{\varepsilon_1\mu_1 - (k_x^2/k_0^2)}}{\varepsilon_1} \tag{B.33}$$

for the isotropic medium, we can write all the Fresnel coefficients for the different cases when the wave is incident from the positive or negative directions as

$$r_{\mathrm{mp}} = \frac{\eta_i - \eta_p}{\eta_i + \eta_p}, \qquad t_{\mathrm{pp}} = \frac{2\eta_i}{\eta_i + \eta_p}, \tag{B.34}$$

$$r'_{\mathrm{mp}} = -\frac{\eta_i - \eta_p}{\eta_i - \eta_m}, \qquad t'_{\mathrm{pp}} = \frac{\eta_p - \eta_m}{\eta_i - \eta_m}, \tag{B.35}$$

$$r_{\mathrm{pm}} = \frac{\eta_i + \eta_m}{\eta_i - \eta_m}, \qquad t_{\mathrm{mm}} = \frac{2\eta_i}{\eta_i - \eta_m}, \tag{B.36}$$

$$r'_{\mathrm{pm}} = -\frac{\eta_i + \eta_m}{\eta_i + \eta_p}, \qquad t'_{\mathrm{mm}} = \frac{\eta_p - \eta_m}{\eta_i + \eta_p}. \tag{B.37}$$

Here the primed coefficients indicate the cases where the wave is incident from the bianisotropic medium onto the isotropic medium and the unprimed coefficients are for the case when the wave is incident from the isotropic medium onto the bianisotropic medium.

The derivation of the dispersion and the Fresnel coefficients for bianisotropic media (even where the two polarizations are not linearly polarized) can be performed in an analogous manner and is left as an exercise to the reader.